

SPDEs with Polynomial Growth Coefficients and Malliavin Calculus Method

Qi Zhang ^a, Huaizhong Zhao ^b

Abstract I

In this paper we study the existence and uniqueness of the $L^2_\rho(\mathbb{R}^d; \mathbb{R}^1) \times L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$ valued solution of backward doubly stochastic differential equations with polynomial growth coefficients using weak convergence, equivalence of norm principle and Wiener-Sobolev compactness arguments. Then we establish a new probabilistic representation of the weak solutions of SPDEs with polynomial growth coefficients through the solutions of the corresponding backward doubly stochastic differential equations (BDSDEs). This probabilistic representation is then used to prove the existence of stationary solution of SPDEs on \mathbb{R}^d via infinite horizon BDSDEs. The convergence of the solution of BDSDE to the solution of infinite horizon BDSDEs is shown to be equivalent to the convergence of the pull-back of the solutions of SPDEs. With this we obtain the stability of the stationary solutions as well.

Keywords: SPDEs with polynomial growth coefficients, probabilistic representation of weak solutions, backward doubly stochastic differential equations, Malliavin derivative, Wiener-Sobolev compactness, stationary solution.

AMS 2000 subject classifications: 60H15, 60H10, 37H10.

1 Introduction

In this paper, we study SPDEs on \mathbb{R}^d with a polynomial growth coefficients of the following type

$$dv(t, x) = [\mathcal{L}v(t, x) + f(x, v(t, x))]dt + g(x, v(t, x))dB_t. \quad (1.1)$$

Here \mathcal{L} is a second order differential operator given by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}. \quad (1.2)$$

^a School of Mathematical Sciences, Fudan University, Shanghai, 200433, China

^b Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK
Emails: qzh@fudan.edu.cn; H.Zhao@lboro.ac.uk

Brownian motion B is a Q -Wiener process with values in a separable Hilbert space U . Denote the countable base of U by $\{e_i\}_{i=1}^\infty$, then $Q \in L(U)$ is a symmetric nonnegative trace class operator such that $Qe_i = \lambda_i e_i$ and $\sum_{i=1}^\infty \lambda_i < \infty$. The coefficients $f : \mathbb{R}^d \times \mathbb{R}^1 \ni (x, v) \mapsto f(x, v) \in \mathbb{R}^1$ is a real-valued function of polynomial growth ($p \geq 2$); $g : \mathbb{R}^d \times \mathbb{R}^1 \ni (x, v) \mapsto g(x, v) \in \mathcal{L}_{U_0}^2(\mathbb{R}^1)$ is a Lipschitz continuous function, where $U_0 = Q^{\frac{1}{2}}(U) \subset U$ is a separable Hilbert space with the norm $\langle u, v \rangle_{U_0} = \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v \rangle_U$ and the complete orthonormal base $\{\sqrt{\lambda_i}e_i\}_{i=1}^\infty$ and $\mathcal{L}_{U_0}^2(\mathbb{R}^1)$ is the space of all Hilbert-Schmidt operators from U_0 to \mathbb{R}^1 with the Hilbert-Schmidt norm.

One of the goals of this article is to study the probabilistic representation to the solution of this equation via the corresponding backward doubly stochastic differential equation (BDSDE). Apart from using the Feynman-Kac formula, the solutions of the backward stochastic differential equations (BSDEs), when they have some regularities (continuous and differentiable in classical sense, or the solutions and their weak derivatives exist in certain weighted $L^q(\mathbb{R}^1, dtdP\rho^{-1}(x)dx) \times L^2(\mathbb{R}^d, dtdP\rho^{-1}(x)dx)$ space) can give a probabilistic representation of the corresponding PDEs. This has been achieved for classical solutions when the coefficients are smooth enough in Pardoux and Peng [21] and for viscosity solution when the coefficients are Lipschitz continuous in [21] and for weak solutions in [2], [3], [28]. When the coefficients are non-Lipschitz, researchers have made some significant progress. In [15], Lepeltier and San Martin assumed that the \mathbb{R}^1 -valued function $f(r, x, y, z)$ satisfies the measurable condition, the y, z linear growth condition and the y, z continuous condition, then they proved the existence of the solution of the corresponding BSDEs. But the uniqueness of the solution failed to be proved since the comparison theorem cannot be used under the non-Lipschitz condition. In Zhang and Zhao [29], we proved the existence and uniqueness of the solution in the space $S^2(\mathbb{R}^1, dtdP\rho^{-1}(x)dx) \times L^2(\mathbb{R}^d, dtdP\rho^{-1}(x)dx)$ to BDSDEs under the monotonicity and linear growth conditions, without assuming Lipschitz condition. We also gave the probabilistic representation of the weak solutions of the corresponding SPDEs. Along the line of the viscosity solution, in [13], Kobylanski was able to solve the BSDE when the coefficients $f(y, z)$ is of quadratic growth in z , with the help of Hopf-Cole transformation. In [20], Pardoux used weak convergence in a finite dimensional space to study the viscosity solution of the PDEs and the corresponding BSDEs when $f(y, z)$ is of polynomial growth in y . As to the weak solutions of PDEs with polynomial growth coefficients and the corresponding BSDEs, the existing methods in BSDE were not adequate to solving the equation. In [30], we developed a new method to BSDEs using Alaoglu weak convergence theorem and Rellich-Kondrachov compactness embedding theorem to get a strongly convergent subsequence in the space $L^2(\mathbb{R}^1, dt\rho^{-1}(x)dx)$, and then using equivalence of norm principle to get the convergent subsequence for BSDEs. We therefore established the correspondence of the solutions of BSDEs and the weak solutions of such PDEs.

To solve the BDSDEs corresponding to SPDE (1.1) when f is of polynomial growth in y , we can still use the Alaoglu weak convergence argument. But the key to make it work is to find a strong convergent subsequence. In the deterministic case, we use the estimate for Sobolev norm of the solutions of the sequence of BSDEs to get a strong convergent subsequence in $L^2(\rho^{-1}(x)dx)$. But this method does not work for the BDSDEs as the subsequence choice may depend on $\omega \in \Omega$. In this paper, we will develop a method using Wiener-Sobolev compactness argument to tackle the compactness problem for BDSDEs. First we estimate the Sobolev norm of the solution as well as the Malliavin derivative to get convergent sequence in $L^2(dtdP_B\rho^{-1}(x)dx)$, then from equivalence of norm principle to pass the compactness to the solutions of BDSDEs in $L^2(dtdP_BdP_W\rho^{-1}(x)dx)$. The Wiener-Sobolev compact embedding theorem is a powerful tool in proving the relatively compactness of a random field. The

random version (independent of time and spatial variables) was obtained in Da Prato, Malliavin and Nualart [7], Peszat [23]. This was extended later by Bally and Sausseureau [4]. This has been extended to the space $C([0, T]; L^2(dPdx))$ and applied to study the existence of an infinite horizon stochastic integral equation arising in the study of random period solution in Feng, Zhao and Zhou [12], Feng and Zhao [11].

Our motivation to study the probabilistic representation is to use it to study the dynamics of the random dynamical system generated by the SPDEs, although the probabilistic representation and weak solution of BDSDEs with polynomial growth coefficients were interesting problems themselves and have their own interest. Stationary solution is one of the central concepts in the study of the long term behaviour of the stochastic dynamical systems generated by SPDEs. It is a pathwise equilibrium which is invariant, over time, along its measurable and P -preserving metric dynamical system $\theta_t : \Omega \rightarrow \Omega$. In deterministic case, it gives the solution of elliptic equation. Due to the nature of the noise that is pumped to the system constantly, the stationary solution is random and changes along times. Therefore the study is rather difficult and no universal method, but important in order to understand the equilibrium and long time behaviour of stochastic systems. There are many works in the literature on the local behaviour of the solutions near a stationary solution, if exists (e.g. Arnold [1], Duan, Lu and Schmalfuss [9], Mohammed, Zhang and Zhao [18], Lian and Lu [16] to name but a few). So the existence is a key to understand complexity of many random dynamical systems. Although there is no universal method applicable to generic problems, researchers have obtained many results to a variety of SPDEs e.g. Sinai [25], [26], Mattingly [17], E, Khanin, Mazel and Sinai [10], Caraballo, Kloeden and Schmalfuss [6], Zhang and Zhao [28], [29]. The cases of non-dissipative stochastic differential equations and SPDEs with additive noise has been obtained in Feng, Zhao and Zhou [12], Feng and Zhao [11]. In applications, stationary solutions also appear in many other real world problems, e.g. in the interpolation of data and image processing, the stationary solution of the stochastic parabolic infinity Laplacian equation gives the final restored image of the image processing in a random model (Wei and Zhao [27]). Note that in the corresponding deterministic model, the elliptic infinity Laplacian equation gives the final restored image as the limit of the solution of the infinity Laplacian equation (Caselles, Morel and Sbert [5]). In this paper, we will solve the infinite horizon BDSDEs with the polynomial growth coefficients and therefore obtain the stationary solutions of SPDEs (1.1). We will also prove that the convergence of the solution of BDSDE to the solution of infinite horizon BDSDEs is equivalent to the convergence of the pull-back of the solutions of SPDEs. Therefore, we obtain the convergence of the solution with a class of initial condition h converges to the stationary solution.

2 Preliminaries and definitions

We will study the weak solutions of the SPDE (1.1) and the corresponding BDSDE in a Hilbert space (ρ -weighted $L^2(dx)$ space). Utilizing this correspondence, we will give the probabilistic representation of the weak solution of SPDE (1.1) on finite horizon with a given initial value and find the stationary solution of SPDE (1.1).

For this purpose, we study first backward SPDE. Let (Ω, \mathcal{F}, P) be a probability space and \hat{B} , W be mutually independent Brownian motions in U and \mathbb{R}^d , respectively. In Section 6, we choose \hat{B} to

be time reversal Brownian motion of B so establish connection with forward SPDEs, especially its stationary solution. Here we consider general Brownian motion \hat{B} .

We first consider the following backward SPDE:

$$\begin{aligned} u(t, x) = & h(x) + \int_t^T [\mathcal{L}u(s, x) + f(s, x, u(s, x))]ds \\ & - \int_t^T g(s, x, u(s, x))d^\dagger \hat{B}_s, \quad 0 \leq t \leq T. \end{aligned} \quad (2.1)$$

Here \mathcal{L} is given by (1.2) with $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $a = \sigma\sigma^* : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$. Assume $h : \mathbb{R}^d \rightarrow \mathbb{R}^1$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ and $g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \rightarrow \mathcal{L}_{U_0}^2(\mathbb{R}^1)$ are measurable. The stochastic integral $\int_t^T g(s, x, u(s, x))d^\dagger \hat{B}_s$ is a backward stochastic integral which will be made clear later.

Denote by $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ the space of measurable functions $l : \mathbb{R}^d \rightarrow \mathbb{R}^1$ such that $\int_{\mathbb{R}^d} l^2(x)\rho^{-1}(x)dx < \infty$. Define the inner product

$$\langle l_1, l_2 \rangle = \int_{\mathbb{R}^d} l_1(x)l_2(x)\rho^{-1}(x)dx,$$

then $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ is a Hilbert space. Here $\rho(x) = (1 + |x|)^q$, $q > d + 8p$, is a weight function and p is given in Condition (H.1). Similarly, denote by $L_\rho^k(\mathbb{R}^d; \mathbb{R}^1)$, $k \geq 2$, the weighted L^k space with the norm $\|l\|_{L_\rho^k(\mathbb{R}^d)} = (\int_{\mathbb{R}^d} l^k(x)\rho^{-1}(x)dx)^{\frac{1}{k}}$. It is easy to see that $\rho(x) : \mathbb{R}^d \rightarrow \mathbb{R}^1$ is a continuous positive function satisfying $\int_{\mathbb{R}^d} |x|^{8p}\rho^{-1}(x)dx < \infty$. We can consider more general $\rho(x)$ as in [2] and all the results of this paper still hold. But this is not the purpose of this paper. Note that due to the polynomial growth of f , we need $u(t, \cdot) \in L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)$ for $t \in [0, T]$.

Now define $X_s^{t,x}$ to be the solution of the following stochastic differential equations for any given $t \geq 0$ and $x \in \mathbb{R}^d$:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, & s \geq t, \\ X_s^{t,x} = x, & 0 \leq s < t. \end{cases} \quad (2.2)$$

The BDSDE associated with SPDE (2.1) is

$$\begin{aligned} Y_s^{t,x} = & h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x})dr \\ & - \int_s^T g(r, X_r^{t,x}, Y_r^{t,x})d^\dagger \hat{B}_r - \int_s^T \langle Z_r^{t,x}, dW_r \rangle, \quad 0 \leq t \leq s \leq T. \end{aligned} \quad (2.3)$$

It is well known that \hat{B} has the following expansion ([8]): for each r ,

$$\hat{B}_r = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \hat{\beta}_j(r) e_j, \quad (2.4)$$

where

$$\hat{\beta}_j(r) = \frac{1}{\sqrt{\lambda_j}} \langle \hat{B}_r, e_j \rangle_U, \quad j = 1, 2, \dots$$

are mutually independent real-valued Brownian motions on (Ω, \mathcal{F}, P) and the series (2.4) is convergent in $L^2(\Omega, \mathcal{F}, P)$. Set $g_j \triangleq g\sqrt{\lambda_j}e_j : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$, then BDSDE (2.3) is equivalent to

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x})dr \\ &\quad - \sum_{j=1}^{\infty} \int_s^T g_j(r, X_r^{t,x}, Y_r^{t,x}) d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x}, dW_r \rangle, \quad 0 \leq t \leq s \leq T. \end{aligned} \quad (2.5)$$

For the convenience of readers, we need to recall the definitions of weak solutions of SPDEs and the $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$ valued solutions of BDSDEs. Denote by \mathcal{N} the class of P -null sets of \mathcal{F} and let

$$\mathcal{F}_{s,T} \triangleq \mathcal{F}_{s,T}^{\hat{B}} \bigvee \mathcal{F}_{t,s}^W, \quad \text{for } 0 \leq t \leq s \leq T, \quad \mathcal{F}_s \triangleq \mathcal{F}_{s,\infty}^{\hat{B}} \bigvee \mathcal{F}_{t,s}^W, \quad \text{for } 0 \leq t \leq s,$$

where for any process $(\eta_s)_{s \geq 0}$, $\mathcal{F}_{t,s}^\eta = \sigma\{\eta_r - \eta_t; 0 \leq t \leq r \leq s\} \bigvee \mathcal{N}$, $\mathcal{F}_{s,\infty}^\eta = \bigvee_{T \geq s} \mathcal{F}_{s,T}^\eta$.

First recall

Definition 2.1 (*Definitions 2.1, [28]*) Let \mathbb{S} be a separable Banach space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ -field \mathcal{S} and $q \geq 2$, $K > 0$. We denote by $M^{q,-K}([t, \infty); \mathbb{S})$ the set of $\mathcal{B}([t, \infty)) \otimes \mathcal{F}/\mathcal{S}$ measurable random processes $\{\phi(s)\}_{s \geq t}$ with values in \mathbb{S} satisfying

- (i) $\phi(s) : \Omega \longrightarrow \mathbb{S}$ is \mathcal{F}_s measurable for $s \geq t$;
- (ii) $E[\int_t^\infty e^{-Ks} \|\phi(s)\|_{\mathbb{S}}^q ds] < \infty$.

Also we denote by $S^{q,-K}([t, \infty); \mathbb{S})$ the set of $\mathcal{B}([t, \infty)) \otimes \mathcal{F}/\mathcal{S}$ measurable random processes $\{\psi(s)\}_{s \geq t}$ with values in \mathbb{S} satisfying

- (i) $\psi(s) : \Omega \longrightarrow \mathbb{S}$ is \mathcal{F}_s measurable for $s \geq t$ and $\psi(\cdot, \omega)$ is a.s. continuous;
- (ii) $E[\sup_{s \geq t} e^{-Ks} \|\psi(s)\|_{\mathbb{S}}^q] < \infty$.

If we replace time interval $[t, \infty)$ by $[t, T]$ in the above definition, we denote the spaces by $M^{q,0}([t, T]; \mathbb{S})$ and $S^{q,0}([t, T]; \mathbb{S})$, respectively. Note that here e^{-Ks} does not play role as T is finite, so we can always take $K = 0$.

For the backward stochastic integral, let $\{g(s)\}_{s \geq 0}$ be a stochastic process with values in $\mathcal{L}_{U_0}^2(H)$ such that $g(s)$ is \mathcal{F}_s measurable for any $s \geq 0$ and locally square integrable, i.e. for any $0 \leq a \leq b < \infty$, $\int_a^b \|g(s)\|_{\mathcal{L}_{U_0}^2(H)}^2 ds < \infty$ a.s. Since \mathcal{F}_s is a backward filtration with respect to \hat{B} , so from the one-dimensional backward Itô's integral and relation with forward integral, for $0 \leq T \leq T'$, we have

$$\int_t^T \sqrt{\lambda_j} \langle g(s)e_j, f_k \rangle d^\dagger \hat{\beta}_j(s) = - \int_{T'-T}^{T'-t} \sqrt{\lambda_j} \langle g(T'-s)e_j, f_k \rangle d\beta_j(s), \quad j, k = 1, 2, \dots$$

where $\beta_j(s) = \hat{\beta}_j(T'-s) - \hat{\beta}_j(T')$, $j = 1, 2, \dots$, and so $B_s = \hat{B}_{T'-s} - \hat{B}_{T'}$. Here $\{f_k\}$ is the complete orthonormal basis in H . From approximation theorem of the stochastic integral in a Hilbert space (cf. [8]), we have

$$\int_{T'-T}^{T'-t} g(T'-s) dB_s = \sum_{j,k=1}^{\infty} \int_{T'-T}^{T'-t} \sqrt{\lambda_j} \langle g(T'-s)e_j, f_k \rangle d\beta_j(s) f_k.$$

Similarly we also have

$$\int_t^T g(s) d^\dagger \hat{B}_s = \sum_{j,k=1}^{\infty} \int_t^T \sqrt{\lambda_j} \langle g(s) e_j, f_k \rangle d^\dagger \hat{\beta}_j(s) f_k.$$

It turns out that

$$\int_t^T g(s) d^\dagger \hat{B}_s = - \int_{T'-T}^{T'-t} g(T' - s) dB_s \quad \text{a.s.}$$

Definition 2.2 A function u is called a weak solution of SPDE (2.1) if $(u, \sigma^* \nabla u) \in L^{2p}([0, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times L^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and for an arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$,

$$\begin{aligned} & \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} h(x) \varphi(x) dx - \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u)(s, x) (\sigma^* \nabla \varphi)(x) dx ds \\ & - \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div}((b - \tilde{A})\varphi)(x) dx ds \\ & = \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x)) \varphi(x) dx ds - \int_t^T \int_{\mathbb{R}^d} g(s, x, u(s, x)) \varphi(x) dx d^\dagger \hat{B}_s, \quad t \in [0, T]. \end{aligned} \quad (2.6)$$

Here $\tilde{A}_j \triangleq \frac{1}{2} \sum_{i=1}^d \frac{\partial a_{ij}(x)}{\partial x_i}$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$.

Remark 2.3 The weak solution of forward SPDE (1.1) with initial value $v(0, \cdot)$ can be defined similarly. We also represent it in a form like $v(t, \cdot, v(0, \cdot))$ to emphasize its dependence on the initial value $v(0, \cdot)$, when it is necessary.

We then give the definition for the $L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$ valued solution of BDSDE (2.3).

Definition 2.4 A pair of processes $(Y_s^{t,x}, Z_s^{t,x})$ is called a solution of BDSDE (2.3) if $(Y_s^{t,\cdot}, Z_s^{t,\cdot}) \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (2.3) for a.e. $x \in \mathbb{R}^d$ a.s.

Remark 2.5 Due to the density of $C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$, we have, for a.e. x , $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies (2.3) is equivalent to for an arbitrary $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$, $(Y_s^{t,x}, Z_s^{t,x})$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} Y_s^{t,x} \varphi(x) dx &= \int_s^T \int_{\mathbb{R}^d} f(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx dr - \int_s^T \int_{\mathbb{R}^d} g(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx d^\dagger \hat{B}_r \\ &- \int_s^T \left\langle \int_{\mathbb{R}^d} Z_r^{t,x} \varphi(x) dx, dW_r \right\rangle \quad P - \text{a.s.} \end{aligned}$$

To find the stationary solution of SPDE (1.1), we need to consider its corresponding infinite horizon BDSDE:

$$\begin{aligned} e^{-Ks} Y_s^{t,x} &= \int_s^\infty e^{-Kr} f(X_r^{t,x}, Y_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr \\ &- \int_s^\infty e^{-Kr} g(X_r^{t,x}, Y_r^{t,x}) d^\dagger \hat{B}_r - \int_s^\infty e^{-Kr} \langle Z_r^{t,x}, dW_r \rangle. \end{aligned} \quad (2.7)$$

For the existence and uniqueness of the solution, we can study a more general form of the above infinite horizon BDSDE with time variable dependent coefficients and $X_s^{t,x}$ is still the flow generated by (2.2):

$$e^{-Ks} Y_s^{t,x} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr$$

$$- \int_s^\infty e^{-Kr} g(r, X_r^{t,x}, Y_r^{t,x}) d^\dagger \hat{B}_r - \int_s^\infty e^{-Kr} \langle Z_r^{t,x}, dW_r \rangle. \quad (2.8)$$

Here $f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1$, $g : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^1 \longrightarrow \mathcal{L}_{U_0}^2(\mathbb{R}^1)$.

Definition 2.6 A pair of processes $(Y_s^{t,x}, Z_s^{t,x})$ is called a solution of BDSDE (2.8) if $(Y^{\cdot, \cdot}, Z^{\cdot, \cdot}) \in S^{2p, -K} \cap M^{2p, -K}([t, \infty); L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2, -K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (2.8) for a.e. $x \in \mathbb{R}^d$ a.s.

For arbitrary $T \geq t$, the general connection between the solution of BDSDE (2.7) and stationary solution of SPDE (1.1) was established in Zhang and Zhao [29]. Such a connection is proved independent of T at which B is reversed to \hat{B} as shown in [28], [29]. Therefore to find the solution of the infinite horizon BDSDE (2.7) is the key to construct the stationary solution of SPDE (1.1).

For $k \geq 0$, denote by $C_{l,b}^k$ the set of C^k -functions whose partial derivatives up to k th order are bounded, but the functions themselves need not be bounded, otherwise if the functions themselves are also bounded, we denote this subspace of $C_{l,b}^k$ by C_b^k . The following generalized equivalence of norm principle is an extension of equivalence of norm principle given in [14], [3], [2] to the case when φ and Ψ are random.

Lemma 2.7 (generalized equivalence of norm principle [28]) Let X be the diffusion process defined in (2.2) with $b \in C_{b,l}^2(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^3(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$. If $s \in [t, T]$, $\varphi : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$ is independent of the σ -field $\mathcal{F}_{t,s}^W$ and $\varphi \rho^{-1} \in L^1(\Omega \times \mathbb{R}^d)$, then there exist two constants $c > 0$ and $C > 0$ s.t.

$$cE\left[\int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx\right] \leq E\left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho^{-1}(x) dx\right] \leq CE\left[\int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx\right].$$

Moreover if $\Psi : \Omega \times [t, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^1$, $\Psi(s, \cdot)$ is independent of $\mathcal{F}_{t,s}^W$ and $\Psi \rho^{-1} \in L^1(\Omega \times [t, T] \times \mathbb{R}^d)$, then

$$\begin{aligned} cE\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho^{-1}(x) dx ds\right] &\leq E\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, X_s^{t,x})| \rho^{-1}(x) dx ds\right] \\ &\leq CE\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho^{-1}(x) dx ds\right]. \end{aligned}$$

In the process of obtaining the stationary solution of SPDE (1.1), the proof of existence and uniqueness of solution to BDSDE (2.3) is a crucial and challenging step. For this, we will start from studying BDSDE (2.3) with finite dimensional noise in next two sections:

$$\begin{aligned} Y_s^{t,x,N} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,N}) dr \\ &\quad - \sum_{j=1}^N \int_s^T g_j(r, X_r^{t,x}, Y_r^{t,x,N}) d^\dagger \hat{\beta}_j(r) - \int_s^T \langle Z_r^{t,x,nN}, dW_r \rangle, \quad 0 \leq t \leq s \leq T. \end{aligned} \quad (2.9)$$

Actually, when N tends to infinity, the solution of BDSDE (2.9) converges to the solution of BDSDE (2.5) which is equivalent to BDSDE (2.3).

The rest of this paper is organized as follows. In Sections 3, we consider approximating BDSDE with Lipschitz coefficients and then use Alaoglu lemma to get a weakly convergent subsequence. We further utilize the equivalence of norm principle and Malliavin derivatives to get a strongly convergent subsequence and prove the existence and uniqueness of the solution to BDSDE (2.9) in Section 4. In

Section 5 we prove that BDSDE (2.3), its corresponding backward SPDE (2.1) and hence, by variable changes, SPDE (1.1), have a unique weak solution. The stationary properties of solutions of BDSDE (2.7) and SPDE (1.1) are shown in Section 6 after proving the existence and uniqueness for solution of infinite horizon BDSDE (2.8).

3 The weak convergence

In this section, we consider BDSDE (2.9) which can be written as

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}) dr - \int_s^T \langle g(r, X_r^{t,x}, Y_r^{t,x}), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle, \quad (3.1)$$

for $0 \leq t \leq s \leq T$. Here $\hat{B} = (\beta_1, \beta_2, \dots, \beta_l)$ is a l -dimensional Brownian motion. We assume

- (H.1). There exists a constant $p \geq 2$ and a function $f_0 : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ with $\int_0^T \int_{\mathbb{R}^d} |f_0(s, x)|^{8p} \rho^{-1}(x) dx ds < \infty$ s.t. for any $s \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^1$, $|f(s, x, y)| \leq L(|f_0(s, x)| + |y|^p)$ and $|\partial_y f(s, x, y)| \leq L(1 + |y|^{p-1})$.
- (H.2). For the above $p \geq 2$, and for any $s, s_1, s_2 \in [0, T]$, $x, x_1, x_2 \in \mathbb{R}^d$, $y, y_1, y_2 \in \mathbb{R}^1$,

$$\begin{aligned} |f(s, x_1, y) - f(s, x_2, y)| &\leq L(1 + |y|^p)|x_1 - x_2|, \\ |g(s_1, x_1, y_1) - g(s_2, x_2, y_2)| &\leq L(|s_1 - s_2| + |x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

Moreover, $\partial_y f$, $\partial_y g$ exist and satisfy

$$\begin{aligned} |\partial_y f(s, x_1, y) - \partial_y f(s, x_2, y)| &\leq L(1 + |y|^{p-1})|x_1 - x_2|, \\ |\partial_y f(s, x, y_1) - \partial_y f(s, x, y_2)| &\leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|, \\ |\partial_y g(s, x, y)| &\leq L, \\ |\partial_y g(s, x_1, y_1) - \partial_y g(s, x_2, y_2)| &\leq L(|x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

- (H.3). There exists a constant $\mu \in \mathbb{R}^1$ s.t. for any $s \in [0, T]$, $x \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^1$,

$$(y_1 - y_2)(f(s, x, y_1) - f(s, x, y_2)) \leq \mu |y_1 - y_2|^2.$$

- (H.4). For the above $p \geq 2$, $\int_{\mathbb{R}^d} |h(x)|^{8p} \rho^{-1}(x) dx < \infty$ and $E[\int_{\mathbb{R}^d} |h(X_T^{t',x}) - h(X_T^{t,x})|^q \rho^{-1}(x) dx] \leq L|t' - t|^{\frac{q}{2}}$ for any $2 \leq q \leq 8p$ and X defined by (2.2).

- (H.5). The diffusion coefficients $b \in C_{l,b}^2(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^3(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$.

- (H.6). The matrix $\sigma(x)$ is uniformly elliptic, i.e. there exists a constant $\varepsilon > 0$ s.t. $\sigma \sigma^*(x) \geq \varepsilon I_d$.

Remark 3.1 (i) In (H.1) and (H.4), the power $8p$ is determined by the estimates in Theorem 4.3.
(ii) Condition (H.4) is weaker than Lipschitz condition of h . This assumption is not for the sake to weaken the Lipschitz condition of h . When we consider the stationary solution in Section 6, we cannot prove that the stationary solution is Lipschitz in x , but we can prove that it satisfies Condition (H.4) in Lemma 6.8.

(iii) The smooth condition (H.5) guarantees the existence of the flow of diffeomorphism. This is essential in the equivalence of norm principle (Lemma 2.7), which together with the uniform ellipticity

condition (H.6) implies the equivalence of the norm between the solution of SPDE and the solution of BDSDE. See (4.2) in Section 4.

From (H.2) and the fact that $\int_{\mathbb{R}^d} |x|^{8p} \rho^{-1}(x) dx < \infty$, we have

$$\sup_{s \in [0, T]} \int_{\mathbb{R}^d} |g(s, x, 0)|^{8p} \rho^{-1}(x) dx < \infty. \quad (3.2)$$

It is easy to see that $(Y_s^{t,x}, Z_s^{t,x})$ solves BDSDE (3.1) for a.e. $x \in \mathbb{R}^d$ if and only if $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x}) = (e^{\mu s} Y_s^{t,x}, e^{\mu s} Z_s^{t,x})$ solves the following BDSDE:

$$\tilde{Y}_s^{t,x} = \tilde{h}(X_T^{t,x}) + \int_s^T \tilde{f}(r, X_r^{t,x}, \tilde{Y}_r^{t,x}) dr - \int_s^T \langle \tilde{g}(r, X_r^{t,x}, \tilde{Y}_r^{t,x}), d^\dagger \hat{B}_r \rangle - \int_s^T \langle \tilde{Z}_r^{t,x}, dW_r \rangle, \quad (3.3)$$

where $\tilde{h}(x) = e^{\mu T} h(x)$, $\tilde{f}(r, x, y) = e^{\mu r} f(r, x, e^{-\mu r} y) - \mu y$ and $\tilde{g}(r, x, y) = e^{\mu r} g(r, x, e^{-\mu r} y)$. We can verify that \tilde{h} , \tilde{f} and \tilde{g} satisfy Conditions (H.1)–(H.4) with a possibly different constant L . But, by Condition (H.3), for any $s \in [0, T]$, $y_1, y_2 \in \mathbb{R}^1$, $x \in \mathbb{R}^d$,

$$\begin{aligned} & (y_1 - y_2)(\tilde{f}(s, x, y_1) - \tilde{f}(s, x, y_2)) \\ &= e^{2\mu s} (e^{-\mu s} y_1 - e^{-\mu s} y_2) (f(s, x, e^{-\mu s} y_1) - f(s, x, e^{-\mu s} y_2)) - \mu(y_1 - y_2)(y_1 - y_2) \leq 0. \end{aligned}$$

Since $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ if and only if $(\tilde{Y}^{t,\cdot}, \tilde{Z}^{t,\cdot}) \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$, so we claim $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of BDSDE (3.1) if and only if $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x})$ is the solution of BSDE (3.3). Therefore we can replace, without losing any generality, Condition (H.3) by

(H.3)*. For any $s \in [0, T]$, $x \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^1$,

$$(y_1 - y_2)(f(s, x, y_1) - f(s, x, y_2)) \leq 0.$$

The main task of Sections 3 and 4 is to prove the following theorem about the existence and uniqueness of the solution of BDSDE (3.1).

Theorem 3.2 *Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), BDSDE (3.1) has a unique solution $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{8p,0}([t, T]; L_\rho^{8p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$.*

For this, a sequence of BDSDEs with linear growth coefficients are constructed. Assume that f in BDSDE (3.1) satisfy Conditions (H.1)–(H.2) and (H.3)*. Firstly, for each $n \in \mathbb{N}$, define

$$f_n(s, x, y) = f(s, x, \Pi_n(y)) + \partial_y f(s, x, \frac{n}{|y|} y) (y - \frac{n}{|y|} y) I_{\{|y| > n\}},$$

where $\Pi_n(y) = \frac{\inf(n, |y|)}{|y|} y$. Obviously, for any $s \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^1$,

$$f_n(s, x, y) \longrightarrow f(s, x, y), \quad \text{as } n \rightarrow \infty,$$

and it is easy to check that f_n satisfies the following conditions:

(H.1)'. For any $s \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^1$, $|f_n(s, x, y)| \leq L(|f_0(s, x)| + (2(n \wedge |y|)^{p-1} + 1)|y|)$ and $|\partial_y f_n(s, x, y)| \leq L(1 + |y|^{p-1})$.

(H.2)'. For any $s \in [0, T]$, $x, x_1, x_2 \in \mathbb{R}^d$, $y, y_1, y_2 \in \mathbb{R}^1$,

$$\begin{aligned} |f_n(s, x_1, y) - f_n(s, x_2, y)| &\leq 3L(1 + |y|^p)|x_1 - x_2|, \\ |\partial_y f_n(s, x_1, y) - \partial_y f_n(s, x_2, y)| &\leq L(1 + |y|^{p-1})|x_1 - x_2|, \\ |\partial_y f_n(s, x, y_1) - \partial_y f_n(s, x, y_2)| &\leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|. \end{aligned}$$

(H.3)'. For any $s \in [0, T]$, $x \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^1$,

$$(y_1 - y_2)(f_n(s, x, y_1) - f_n(s, x, y_2)) \leq 0.$$

We then study the following BDSDE with coefficient f_n :

$$Y_s^{t,x,n} = h(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, Y_r^{t,x,n})dr - \int_s^T \langle g(r, X_r^{t,x}, Y_r^{t,x,n}), d\hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle. \quad (3.4)$$

Notice that the coefficients h , f_n and g satisfy Conditions (H.1)'–(H.3)', (H.2), (H.4). Hence by Theorems 2.2 and 2.3 in [29], we have the following proposition:

Proposition 3.3 ([29]) *Under the conditions of Theorem 3.2, BDSDE (3.4) has a unique solution $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. If we define $Y_t^{t,x,n} = u_n(t, x)$, then $u_n(t, x)$ is the unique strong solution of the following SPDE*

$$u_n(t, x) = h(x) + \int_t^T \{\mathcal{L}u_n(s, x) + f_n(s, x, u_n(s, x))\}ds - \int_t^T \langle g(s, x, u_n(s, x)), d^\dagger \hat{B}_s \rangle, \quad (3.5)$$

for $0 \leq t \leq T$. Moreover,

$$u_n(s, X_s^{t,x}) = Y_s^{t,x,n}, \quad (\sigma^* \nabla u_n)(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

The key is to pass the limits in (3.4) and (3.5) in some desired sense. For this, we need some estimates.

Lemma 3.4 *Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), if $(Y^{t,\cdot,n}, Z^{t,\cdot,n})$ is the solution of BDSDE (3.4), then we have for any $2 \leq m \leq 8p$,*

$$\begin{aligned} &\sup_n E[\sup_{s \in [t, T]} \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^m \rho^{-1}(x) dx] + \sup_n E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^m \rho^{-1}(x) dx ds] \\ &+ \sup_n E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^{m-2} |Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds] + \sup_n E[(\int_t^T \int_{\mathbb{R}^d} |Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds)^{\frac{m}{2}}] < \infty. \end{aligned}$$

The proof of the lemma follows some standard Itô's formula computation. So it is omitted here.

Taking $m = 2$ in Lemma 3.4, we know

$$\sup_n E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^2 \rho^{-1}(x) dx ds] + \sup_n E[\int_t^T \int_{\mathbb{R}^d} |Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds] < \infty.$$

Define $U_s^{t,x,n} = f_n(s, X_s^{t,x}, Y_s^{t,x,n})$ and $V_s^{t,x,n} = g(s, X_s^{t,x}, Y_s^{t,x,n})$, $s \geq t$. Using Lemma 3.4 again, we also have

$$\sup_n E[\int_t^T \int_{\mathbb{R}^d} (|Y_s^{t,x,n}|^2 + |Z_s^{t,x,n}|^2 + |U_s^{t,x,n}|^2 + |V_s^{t,x,n}|^2) \rho^{-1}(x) dx ds]$$

$$\begin{aligned}
&\leq \sup_n C_p E \left[\int_t^T \int_{\mathbb{R}^d} (1 + |f_0(s, X_s^{t,x})|^2 + |g(s, X_s^{t,x}, 0)|^2 + |Y_s^{t,x,n}|^{2p} + |Z_s^{t,x,n}|^2) \rho^{-1}(x) dx ds \right] \\
&< \infty.
\end{aligned} \tag{3.6}$$

Here and in the following C_p is a generic constant. Then, according to the Alaoglu lemma, we know that there exists a subsequence, still denoted by $(Y^{t,\cdot,n}, Z^{t,\cdot,n}, U^{t,\cdot,n}, V^{t,\cdot,n})$, converging weakly to a limit $(Y^{t,\cdot}, Z^{t,\cdot}, U^{t,\cdot}, V^{t,\cdot})$ in $L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1 \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^l)$, or equivalently $L^2(\Omega \times [t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^l))$. Now we take the weak limit in $L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$ to BDSDEs (3.4), we can verify that $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, V_s^{t,x})$ satisfies the following BDSDE:

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T U_r^{t,x} dr - \int_s^T \langle V_r^{t,x}, d\hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle. \tag{3.7}$$

For this, we will check the weak convergence term by term. The weak convergence of $Y_s^{t,x,n}$ is deduced by the definition of $Y_s^{t,x}$. We check the weak convergence of $\int_s^T U_r^{t,x,n} dr$. Let $\eta \in L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$. Noticing $\int_t^T \sup_n E[\int_s^T \int_{\mathbb{R}^d} |U_r^{t,x,n}|^2 \rho^{-1}(x) dx dr] ds < \infty$ due to (3.6), by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&|E[\int_t^T \int_{\mathbb{R}^d} \int_s^T (U_r^{t,x,n} - U_r^{t,x}) dr \eta(s, x) \rho^{-1}(x) dx ds]| \\
&\leq \int_t^T |E[\int_s^T \int_{\mathbb{R}^d} (U_r^{t,x,n} - U_r^{t,x}) \eta(s, x) \rho^{-1}(x) dx dr]| ds \longrightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

To prove the weak convergence of $\int_s^T \langle V_r^{t,x,n}, d\hat{B}_r \rangle$, first note that for fixed s and x , $\eta(s, x) \in L^2(\Omega; \mathbb{R}^1)$, hence there exist $\psi \in L^2(\Omega \times [t, T] \times \mathbb{R}^d \times [t, T]; \mathbb{R}^l)$ and $\phi \in L^2(\Omega \times [t, T] \times \mathbb{R}^d \times [t, T]; \mathbb{R}^d)$ s.t. $\eta(s, x) = E[\eta(s, x)] + \int_t^T \langle \psi(s, x, r), d\hat{B}_r \rangle + \int_t^T \langle \phi(s, x, r), dW_r \rangle$. Noticing that for a.e. $s \in [t, T]$, $\psi(s, \cdot, \cdot) \in L^2(\Omega \times \mathbb{R}^d \times [t, T]; \mathbb{R}^l)$, $\phi(s, \cdot, \cdot) \in L^2(\Omega \times \mathbb{R}^d \times [t, T]; \mathbb{R}^d)$ and $\int_t^T \sup_n E[\int_s^T \int_{\mathbb{R}^d} |V_r^{t,x,n}|^2 \rho^{-1}(x) dx dr] ds < \infty$, by Lebesgue's dominated convergence theorem again, we obtain

$$\begin{aligned}
&|E[\int_t^T \int_{\mathbb{R}^d} \int_s^T \langle V_r^{t,x,n} - V_r^{t,x}, d\hat{B}_r \rangle \eta(s, x) \rho^{-1}(x) dx ds]| \\
&= |\int_t^T \int_{\mathbb{R}^d} E[\int_s^T \langle V_r^{t,x,n} - V_r^{t,x}, \psi(s, x, r) \rangle dr] \rho^{-1}(x) dx ds| \\
&\leq \int_t^T |E[\int_s^T \int_{\mathbb{R}^d} \langle V_r^{t,x,n} - V_r^{t,x}, \psi(s, x, r) \rangle \rho^{-1}(x) dx dr]| ds \longrightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

For the weak convergence of last term, we can deduce similarly that

$$\begin{aligned}
&|E[\int_t^T \int_{\mathbb{R}^d} \int_s^T \langle Z_r^{t,x,n} - Z_r^{t,x}, dW_r \rangle \eta(s, x) \rho^{-1}(x) dx ds]| \\
&\leq \int_t^T |E[\int_s^T \int_{\mathbb{R}^d} \langle Z_r^{t,x,n} - Z_r^{t,x}, \phi(s, x, r) \rangle \rho^{-1}(x) dx dr]| ds \longrightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Needless to say, if we can show that BDSDE (3.4) converges weakly to BDSDE (3.1) as $n \rightarrow \infty$, then we can say $(Y_s^{t,x}, Z_s^{t,x})$ is a solution of BDSDE (3.1). The key is to prove that $U_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x})$ and $V_s^{t,x} = g(s, X_s^{t,x}, Y_s^{t,x})$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. However, the weak convergence of Y^n, U^n ,

V^n and Z^n is far from enough for this purpose. The real difficulty in this analysis is to establish the strong convergence of Y^n and Z^n , at least along a subsequence.

4 The strong convergence

To obtain the strongly convergent subsequence of Y^n and Z^n , we need to estimate the Malliavin derivatives to prove the relative compactness of Y_n first. Let \mathcal{O} be a bounded domain in \mathbb{R}^d . Denote $C_c^k(\mathcal{O})$ the class of k -times differentiable functions which have a compact support inside \mathcal{O} . For $\varphi \in C_c^k(\mathcal{O})$, we define $v^\varphi(s, \omega) = \int_{\mathcal{O}} v(s, x, \omega) \varphi(x) dx$. The following theorem proved in Bally and Sausseureau [4] can be regarded as an extension of Rellich-Kondrachov compactness theorem to stochastic case. This kind of Wiener-Sobolev compactness theorem for time and space independent case was considered by Da Prato and Malliavin [7], Peszat [23]. One extension was given in Feng, Zhao and Zhou [12], Feng and Zhao [11] to replace L^2 norm in two variable by sup norm in the two variables, in order to apply it to infinite horizon stochastic integral equations. For the purpose of this paper, the L^2 norm used by Bally and Sausseureau is enough.

Denote by $C_p^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ such that f and all its partial derivatives have polynomial growth. Let \mathbb{K} be the class of smooth random variables F that is $F = f(W(h_1), \dots, W(h_n))$ with $n \in \mathbb{N}$, $h_1, \dots, h_n \in L^2([0, T])$ and $f \in C_p^\infty(\mathbb{R}^n)$. The derivative operator of a smooth random variable F is the stochastic process $\{D_t F, t \in [0, T]\}$ defined by (cf. [19])

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(W(h_1), \dots, W(h_n)) h_i(t).$$

We will denote $\mathbb{D}^{1,2}$ the domain of D in $L^2(\Omega)$, i.e. $\mathbb{D}^{1,2}$ is the closure of \mathbb{K} with respect to the norm

$$\|F\|_{1,2}^2 = E[|F|^2] + E[\|D_t F\|_{L^2([0,T])}^2].$$

Recall

Theorem 4.1 (Theorem 2, [4]) *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of $L^2([0, T] \times \Omega; H^1(\mathcal{O}))$. Suppose that*

- (1) $\sup_n E[\int_0^T \|u_n(s, \cdot)\|_{H^1(\mathcal{O})}^2 ds] < \infty$.
- (2) For all $\varphi \in C_c^k(\mathcal{O})$ and $t \in [0, T]$, $u_n^\varphi(s) \in \mathbb{D}^{1,2}$ and $\sup_n \int_0^T \|u_n^\varphi(s)\|_{\mathbb{D}^{1,2}}^2 ds < \infty$.
- (3) For all $\varphi \in C_c^k(\mathcal{O})$, the sequence $(E[u_n^\varphi])_{n \in \mathbb{N}}$ of $L^2([0, T])$ satisfies
- (3i) For any $\varepsilon > 0$, there exists $0 < \alpha < \beta < T$ s.t.

$$\sup_n \int_{[0,T] \setminus (\alpha, \beta)} |E[u_n^\varphi(s)]|^2 ds < \varepsilon.$$

- (3ii) For any $0 < \alpha < \beta < T$ and $h \in \mathbb{R}^1$ s.t. $|h| < \min(\alpha, T - \beta)$, it holds

$$\sup_n \int_\alpha^\beta |E[u_n^\varphi(s+h)] - E[u_n^\varphi(s)]|^2 ds < C_p |h|.$$

- (4) For all $\varphi \in C_c^k(\mathcal{O})$, the following conditions are satisfied:

- (4i) For any $\varepsilon > 0$, there exists $0 < \alpha < \beta < T$ and $0 < \alpha' < \beta' < T$ s.t.

$$\sup_n E\left[\int_{[0,T]^2 \setminus (\alpha,\beta) \times (\alpha',\beta')} |D_\theta u_n^\varphi(s)|^2 d\theta ds\right] < \varepsilon.$$

(4ii) For any $0 < \alpha < \beta < T$, $0 < \alpha' < \beta' < T$ and $h, h' \in \mathbb{R}^1$ s.t. $\max(|h|, |h'|) < \min(\alpha, \alpha', T - \beta, T - \beta')$, it holds that

$$\sup_n E\left[\int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} u_n^\varphi(s+h') - D_\theta u_n^\varphi(s)|^2 d\theta ds\right] < C_p(|h| + |h'|).$$

Then $(u_n)_{n \in \mathbb{N}}$ is relatively compact in $L^2(\Omega \times [0, T] \times \mathcal{O}; \mathbb{R}^1)$.

Using Theorem 4.1, we can verify that the sequence $u_n(s, x)$ in SPDE (3.5) is relatively compact. In this process, some estimates on the Malliavin derivative of the random variable (Y, Z) w.r.t. Brownian motion B are needed. In what follows, we will need the following results whose proofs are deferred to Section 7. Throughout this paper, Malliavin derivative always refers to Malliavin derivative w.r.t. B unless we say otherwise.

Lemma 4.2 Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), the Malliavin derivative of the solution $(Y_s^{t,x,n}, Z_s^{t,x,n})$ of BDSDE (3.4) exists and satisfies the following linear equation:

$$\begin{cases} D_\theta Y_s^{t,x,n} = g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}) + \int_s^\theta \partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}) D_\theta Y_r^{t,x,n} dr \\ \quad - \int_s^\theta \partial_y g(r, X_r^{t,x}, Y_r^{t,x,n}) D_\theta Y_r^{t,x,n} d^\dagger \hat{B}_r - \int_s^\theta D_\theta Z_r^{t,x,n} dW_r, \quad 0 \leq \theta \leq T, \\ D_\theta Y_s^{t,x,n} = 0, \quad t \leq \theta < s. \end{cases} \quad (4.1)$$

Moreover, for any $2 \leq m \leq 8p$,

$$\begin{aligned} & \sup_n \sup_{\theta \in [t, T]} E\left[\sup_{s \in [t, T]} \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx\right] + \sup_n \sup_{\theta \in [t, T]} E\left[\int_t^T \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx ds\right] \\ & + \sup_n \sup_{\theta \in [t, T]} E\left[\int_t^T \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^{m-2} |D_\theta Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds\right] \\ & + \sup_n \sup_{\theta \in [t, T]} E\left[\left(\int_t^T \int_{\mathbb{R}^d} |D_\theta Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds\right)^{\frac{m}{2}}\right] < \infty. \end{aligned}$$

Proof. From Condition (H.2)' and the results of [4] or [22], it is easy to know that the Malliavin derivative of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ exists and satisfies (4.1). The rest of the proof follows some standard computations using Itô's formula. So it is omitted here. \diamond

Now we are ready to prove the relative compactness of the solutions of SPDE (3.5) in the following theorem. In the proof of Theorem 4.3 and the similar arguments throughout this paper, we will leave out the similar localization procedures as in the proof of Lemma 3.4 when applying generalized Itô's formula, due to the limitation of the length of the paper.

Theorem 4.3 Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), if $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n})$ is the solution of BDSDE (3.4) and \mathcal{O} is a bounded domain in \mathbb{R}^d , then the sequence $u_n(s, x) \triangleq Y_s^{s,x,n}$ is relatively compact in $L^2(\Omega \times [0, T] \times \mathcal{O}; \mathbb{R}^1)$.

Proof. We verify that u_n satisfies Conditions (1)–(4) in Theorem 4.1.

Step 1. We first verify Condition (1). By Conditions (H.5)–(H.6), Lemma 2.7, Proposition 3.3 and Lemma 3.4, we have

$$\begin{aligned} \sup_n E[\int_0^T \|u_n(s, \cdot)\|_{H^1(\mathcal{O})}^2 ds] &\leq C_p \sup_n E[\int_0^T \int_{\mathcal{O}} (|u_n(s, x)|^2 + |\nabla u_n(s, x)|^2) \rho^{-1}(x) dx ds] \\ &\leq C_p \sup_n E[\int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x,n}|^2 + |Z_s^{0,x,n}|^2) \rho^{-1}(x) dx ds] < \infty. \end{aligned} \quad (4.2)$$

Step 2. We then verify Condition (2). It is easy to see that $D_\theta u_n^\varphi(s) = \int_{\mathcal{O}} D_\theta u_n(s, x) \varphi(x) dx$. By Lemma 4.2, $D_\theta u_n(s, x) = D_\theta Y_s^{s,x,n}$ exists. Indeed, we can further prove $u_n^\varphi(s) \in \mathbb{D}^{1,2}$. By stochastic calculus, we have

$$\begin{aligned} \|u_n^\varphi(s)\|_{\mathbb{D}^{1,2}}^2 &\leq C_p E[\int_{\mathbb{R}^d} |u_n(s, x)|^2 \rho^{-1}(x) dx] + C_p E[\int_s^T \int_{\mathbb{R}^d} |D_\theta u_n(s, x)|^2 \rho^{-1}(x) dx d\theta] \\ &\leq C_p E[\int_{\mathbb{R}^d} |Y_s^{0,x,n}|^2 \rho^{-1}(x) dx] + C_p E[\int_s^T \int_{\mathbb{R}^d} |D_\theta Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta] \\ &\leq C_p \sup_n \int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x) dx + C_p \int_0^T \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr \\ &\quad + C_p \int_0^T \int_{\mathbb{R}^d} |g(r, x, 0)|^2 \rho^{-1}(x) dx dr < \infty. \end{aligned} \quad (4.3)$$

Also the right hand side of the above inequality is independent of s and n , so

$$\sup_n \int_0^T \|u_n^\varphi(s)\|_{\mathbb{D}^{1,2}}^2 ds < \infty.$$

Step 3. Let us verify Condition (3). First (3i) follows immediately from (4.3). To see (3ii), assume $h > 0$ without losing any generality. From (3.4) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sup_n \int_\alpha^\beta |E[u_n^\varphi(s+h)] - E[u_n^\varphi(s)]|^2 ds &\leq C_p \sup_n \int_\alpha^\beta E[\int_{\mathbb{R}^d} |u_n(s+h, x) - u_n(s, x)|^2 \rho^{-1}(x) dx] ds \\ &\leq C_p \sup_n \int_\alpha^\beta E[\int_{\mathbb{R}^d} |Y_{s+h}^{0,x,n} - Y_s^{0,x,n}|^2 \rho^{-1}(x) dx] ds \\ &\leq C_p \int_\alpha^\beta \int_s^{s+h} \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr ds \\ &\quad + C_p \sup_n \int_\alpha^\beta \int_s^{s+h} \sup_{r \in [0, T]} (1 + E[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{2p} \rho^{-1}(x) dx]) dr ds \\ &\quad + C_p \sup_n \int_\alpha^\beta \int_s^{s+h} E[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx] dr ds \\ &\quad + C_p \int_\alpha^\beta \int_s^{s+h} \sup_{r \in [0, T]} \int_{\mathbb{R}^d} |g(r, x, 0)|^2 \rho^{-1}(x) dx dr ds. \end{aligned} \quad (4.4)$$

Note that by changing integration order,

$$\sup_n \int_\alpha^\beta \int_s^{s+h} E[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx] dr ds$$

$$\begin{aligned}
&= \sup_n \left(\int_{\alpha}^{\alpha+h} \int_{\alpha}^r E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds dr + \int_{\alpha+h}^{\beta} \int_{r-h}^r E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds dr \right. \\
&\quad \left. + \int_{\beta}^{\beta+h} \int_{r-h}^{\beta} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds dr \right) \\
&= \sup_n \left((r-\alpha) \int_{\alpha}^{\alpha+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr + h \int_{\alpha+h}^{\beta} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \right. \\
&\quad \left. + (\beta+h-r) \int_{\beta}^{\beta+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \right) \\
&= C_p h \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right]. \tag{4.5}
\end{aligned}$$

A similar calculation can be done to $\int_{\alpha}^{\beta} \int_s^{s+h} \int_{\mathbb{R}^d} |f_0(r,x)|^2 \rho^{-1}(x) dx dr ds$. Hence it follows from (4.4) that

$$\begin{aligned}
&\sup_n \int_{\alpha}^{\beta} |E[u_n^{\varphi}(s+h)] - E[u_n^{\varphi}(s)]|^2 ds \\
&\leq C_p h \int_0^T \int_{\mathbb{R}^d} |f_0(r,x)|^2 \rho^{-1}(x) dx dr + C_p h \sup_n \sup_{r \in [0,T]} (1 + E \left[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{2p} \rho^{-1}(x) dx \right]) \\
&\quad + C_p h \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] + C_p h \sup_{r \in [0,T]} \int_{\mathbb{R}^d} |g(r,x,0)|^2 \rho^{-1}(x) dx.
\end{aligned}$$

Also noticing Condition (H.1), (3.2) and Lemma 3.4, we can conclude that (3ii) holds.

Step 4. We now verify Condition (4). For (4i), since by the equivalence of norm principle it turns out that

$$\begin{aligned}
\sup_n \sup_{\theta \in [t,T]} \sup_{s \in [t,T]} E[|D_{\theta} u_n^{\varphi}(s)|^2] &\leq C_p \sup_n \sup_{\theta \in [t,T]} \sup_{s \in [t,T]} E \left[\int_{\mathbb{R}^d} |D_{\theta} u_n(s,x)|^2 \rho^{-1}(x) dx \right] \\
&\leq C_p \sup_n \sup_{\theta \in [t,T]} \sup_{s \in [t,T]} E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_s^{0,x,n}|^2 \rho^{-1}(x) dx \right] < \infty.
\end{aligned}$$

So (4i) follows. To see (4ii), assume without losing any generality that $h, h' > 0$, then

$$\begin{aligned}
&\sup_n E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} |D_{\theta+h} u_n^{\varphi}(s+h') - D_{\theta} u_n^{\varphi}(s)|^2 d\theta ds \right] \\
&\leq C_p \sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta+h} u_n(s,x) - D_{\theta} u_n(s,x)|^2 \rho^{-1}(x) dx d\theta ds \right] \\
&\quad + C_p \sup_n E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta} u_n(s+h',x) - D_{\theta} u_n(s,x)|^2 \rho^{-1}(x) dx d\theta ds \right]. \tag{4.6}
\end{aligned}$$

For the first term on the right hand side of (4.6), by the equivalence of norm principle,

$$\begin{aligned}
&\sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta+h} u_n(s,x) - D_{\theta} u_n(s,x)|^2 \rho^{-1}(x) dx d\theta ds \right] \\
&= \sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta+h} Y_s^{s,x,n} - D_{\theta} Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right]
\end{aligned}$$

$$\leq C_p \sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta+h} Y_s^{0,x,n} - D_\theta Y_s^{0,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right]. \quad (4.7)$$

By BDSDE (4.1) we know that

$$\begin{aligned} (D_{\theta+h} - D_\theta) Y_s^{0,x,n} &= H(\theta, \theta+h) + \int_s^\theta \partial_y f_n(r, X_r^{0,x}, Y_r^{0,x,n})(D_{\theta+h} - D_\theta) Y_r^{0,x,n} dr \\ &\quad - \int_s^\theta \partial_y g(r, X_r^{0,x}, Y_r^{0,x,n})(D_{\theta+h} - D_\theta) Y_r^{0,x,n} d^\dagger \hat{B}_r - \int_s^\theta (D_{\theta+h} - D_\theta) Z_r^{0,x,n} dW_r, \end{aligned}$$

where

$$\begin{aligned} H(\theta, \theta+h) &= g(\theta+h, X_{\theta+h}^{0,x}, Y_{\theta+h}^{0,x,n}) - g(\theta, X_\theta^{0,x}, Y_\theta^{0,x,n}) + \int_\theta^{\theta+h} \partial_y f_n(r, X_r^{0,x}, Y_r^{0,x,n}) D_{\theta+h} Y_r^{0,x,n} dr \\ &\quad - \int_\theta^{\theta+h} \partial_y g(r, X_r^{0,x}, Y_r^{0,x,n}) D_{\theta+h} Y_r^{0,x,n} d^\dagger \hat{B}_r - \int_\theta^{\theta+h} D_{\theta+h} Z_r^{0,x,n} dW_r. \end{aligned}$$

Applying Itô's formula to $e^{Kr} |(D_{\theta+h} - D_\theta) Y_s^{0,x,n}|^2$, we have

$$\begin{aligned} &E \left[\int_{\mathbb{R}^d} |(D_{\theta+h} - D_\theta) Y_s^{0,x,n}|^2 \rho^{-1}(x) dx \right] + E \left[\int_s^\theta \int_{\mathbb{R}^d} |(D_{\theta+h} - D_\theta) Y_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\ &+ E \left[\int_s^\theta \int_{\mathbb{R}^d} |(D_{\theta+h} - D_\theta) Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\ &\leq C_p E \left[\int_{\mathbb{R}^d} |H(\theta, \theta+h)|^2 \rho^{-1}(x) dx \right]. \end{aligned} \quad (4.8)$$

Next we prove that

$$\sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |H(\theta, \theta+h)|^2 \rho^{-1}(x) dx d\theta ds \right] \leq C_p h. \quad (4.9)$$

First note that

$$\begin{aligned} &E \left[\int_{\mathbb{R}^d} |H(\theta, \theta+h)|^2 \rho^{-1}(x) dx \right] \\ &\leq C_p h^2 + C_p E \left[\int_{\mathbb{R}^d} |X_{\theta+h}^{0,x} - X_\theta^{0,x}|^2 \rho^{-1}(x) dx \right] + C_p E \left[\int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_\theta^{0,x,n}|^2 \rho^{-1}(x) dx \right] \\ &\quad + C_p E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |\partial_y f_n(r, X_r^{0,x}, Y_r^{0,x,n})|^2 |D_{\theta+h} Y_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\ &\quad + C_p \int_\theta^{\theta+h} \sup_{s \in [0, T]} \sup_{r \in [0, T]} E \left[\int_{\mathbb{R}^d} |D_s Y_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr + C_p E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |D_s Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right]. \end{aligned} \quad (4.10)$$

We need to estimate each term in the above formula. From (2.2), we have

$$\begin{aligned} &E \left[\int_{\mathbb{R}^d} |X_{\theta+h}^{0,x} - X_\theta^{0,x}|^2 \rho^{-1}(x) dx \right] \\ &\leq C_p E \left[\int_{\mathbb{R}^d} \left(\int_\theta^{\theta+h} |b(X_u^{0,x})| du \right)^2 \rho^{-1}(x) dx \right] + C_p \int_{\mathbb{R}^d} E \left[\int_\theta^{\theta+h} |\sigma(X_u^{0,x})|^2 du \right] \rho^{-1}(x) dx \\ &\leq C_p E \left[\int_{\mathbb{R}^d} h \int_\theta^{\theta+h} (1 + |X_u^{0,x}|)^2 du \rho^{-1}(x) dx \right] + C_p E \left[\int_{\mathbb{R}^d} \int_\theta^{\theta+h} L^2 du \rho^{-1}(x) dx \right] \end{aligned}$$

$$\begin{aligned}
&\leq C_p h \int_{\theta}^{\theta+h} E \left[\int_{\mathbb{R}^d} (1 + |X_u^{0,x}|)^2 \rho^{-1}(x) dx \right] du + C_p h E \left[\int_{\mathbb{R}^d} L^2 \rho^{-1}(x) dx \right] \\
&\leq C_p h.
\end{aligned}$$

By (H.1)', Lemma 3.4 and Lemma 4.2, we have

$$\begin{aligned}
&E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |\partial_y f_n(r, X_r^{0,x}, Y_r^{0,x,n})|^2 |D_{\theta+h} Y_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
&\leq C_p \int_{\theta}^{\theta+h} \sup_n \sup_{\theta \in [0,T]} \sup_{r \in [0,T]} \left(\sqrt{E \left[\int_{\mathbb{R}^d} (1 + |Y_r^{0,x,n}|^{4p-4}) \rho^{-1}(x) dx \right]} \sqrt{E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_r^{0,x,n}|^4 \rho^{-1}(x) dx \right]} \right) dr \\
&\leq C_p h.
\end{aligned}$$

By Lemma 4.2 again, we also have that

$$\int_{\theta}^{\theta+h} \sup_n \sup_{s \in [0,T]} \sup_{r \in [0,T]} E \left[\int_{\mathbb{R}^d} |D_s Y_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \leq C_p h.$$

Hence, from (4.10), to prove (4.9) is reduced to prove

$$\begin{aligned}
&\sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_{\theta}^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \\
&+ \sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |D_s Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \leq C_p h.
\end{aligned} \tag{4.11}$$

From (3.4), we have

$$\begin{aligned}
&E \left[\int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_{\theta}^{0,x,n}|^2 \rho^{-1}(x) dx \right] \\
&\leq C_p E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |f_n(r, X_r^{0,x}, Y_r^{0,x,n})|^2 \rho^{-1}(x) dx dr \right] \\
&+ C_p E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |g(r, X_r^{0,x}, Y_r^{0,x,n})|^2 \rho^{-1}(x) dx dr \right] + C_p E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
&\leq C_p \int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr + C_p \int_{\theta}^{\theta+h} \sup_n \sup_{r \in [0,T]} E \left[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{2p} \rho^{-1}(x) dx \right] dr \\
&+ C_p \int_{\theta}^{\theta+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr + C_p \int_{\theta}^{\theta+h} \sup_{r \in [0,T]} E \left[\int_{\mathbb{R}^d} |g(r, x, 0)|^2 \rho^{-1}(x) dx \right] dr.
\end{aligned} \tag{4.12}$$

A similar calculation of changing the integrations order as in (4.5) leads to

$$\begin{aligned}
&\sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\theta}^{\theta+h} E \left[\int_{\mathbb{R}^d} (|f_0(r, x)|^2 + |Z_r^{0,x,n}|^2) \rho^{-1}(x) dx \right] dr d\theta ds \\
&\leq C_p h \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} (|f_0(r, x)|^2 + |Z_r^{0,x,n}|^2) \rho^{-1}(x) dx dr \right].
\end{aligned}$$

Moreover, by Condition (H.1), Lemma 3.4 and (3.2) we conclude from (4.12) that

$$\sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_{\theta}^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \leq C_p h.$$

Furthermore, by changing the integrations order again and Lemma 4.2, we have

$$\sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_{\theta}^{\theta+h} \int_{\mathbb{R}^d} |D_s Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \leq C_p h.$$

Hence (4.11) follows. So (4.9) holds. Now by (4.7) and (4.8) we can deduce that

$$\sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta+h} u_n(s, x) - D_{\theta} u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds \right] \leq C_p h. \quad (4.13)$$

Now we deal with the second term on the right hand side of (4.6). Notice

$$\begin{aligned} & \sup_n E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta} u_n(s+h', x) - D_{\theta} u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds \right] \\ & \leq \sup_n 2E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta} Y_{s+h'}^{s,x,n} - D_{\theta} Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right] \\ & \quad + \sup_n 2E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta} Y_{s+h'}^{s+h',x,n} - D_{\theta} Y_{s+h'}^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right]. \end{aligned} \quad (4.14)$$

For the first term on the right hand side of (4.14), by (4.1), Lemma 2.7 and the exchange of the integrations, it is easy to see that

$$\begin{aligned} & \sup_n E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta} Y_{s+h'}^{s,x,n} - D_{\theta} Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right] \\ & \leq C_p \sup_n \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} E \left[\int_{\mathbb{R}^d} |D_{\theta} (Y_{s+h'}^{0,x,n} - Y_s^{0,x,n})|^2 \rho^{-1}(x) dx \right] d\theta ds \\ & \leq C_p \sup_{s \in [0, T-h']} \int_s^{s+h'} (1 + \sup_n E \left[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{4p-4} \rho^{-1}(x) dx \right]) dr \\ & \quad + C_p \sup_{s \in [0, T-h']} \int_s^{s+h'} (1 + \sup_n \sup_{\theta \in [0, T]} E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_r^{0,x,n}|^4 \rho^{-1}(x) dx \right]) dr \\ & \quad + C_p h' \sup_n \sup_{\theta \in [0, T]} E \left[\int_0^T \int_{\mathbb{R}^d} |D_{\theta} Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & \leq C_p h'. \end{aligned} \quad (4.15)$$

For the second term on the right hand side of (4.14), firstly from BDSDE (4.1) we know that

$$\begin{aligned} & D_{\theta} (Y_{s+h'}^{s+h',x,n} - Y_{s+h'}^{s,x,n}) \\ & = J(s, s+h') + \int_{s+h'}^{\theta} \partial_y f_n(r, X_r^{s,x}, Y_r^{s,x,n}) D_{\theta} (Y_r^{s+h',x,n} - Y_r^{s,x,n}) dr \\ & \quad - \int_{s+h'}^{\theta} \partial_y g(r, X_r^{s,x}, Y_r^{s,x,n}) D_{\theta} (Y_r^{s+h',x,n} - Y_r^{s,x,n}) d^{\dagger} \hat{B}_r - \int_{s+h'}^{\theta} D_{\theta} (Z_r^{s+h',x,n} - Z_r^{s,x,n}) dW_r, \end{aligned}$$

where

$$\begin{aligned}
J(s, s+h') &= g(\theta, X_\theta^{s+h',x}, Y_\theta^{s+h',x,n}) - g(\theta, X_\theta^{s,x}, Y_\theta^{s,x,n}) \\
&\quad + \int_{s+h'}^\theta (\partial_y f_n(r, X_r^{s+h',x}, Y_r^{s+h',x,n}) - \partial_y f_n(r, X_r^{s,x}, Y_r^{s,x,n})) D_\theta Y_r^{s+h',x,n} dr \\
&\quad - \int_{s+h'}^\theta (\partial_y g(r, X_r^{s+h',x}, Y_r^{s+h',x,n}) - \partial_y g(r, X_r^{s,x}, Y_r^{s,x,n})) D_\theta Y_r^{s+h',x,n} d^\dagger \hat{B}_r.
\end{aligned}$$

Applying Itô's formula to $e^{Kr} |D_\theta(Y_r^{s+h',x,n} - Y_r^{s,x,n})|^2$, we have

$$\begin{aligned}
\sup_n E[\int_{\mathcal{O}} |D_\theta(Y_{s+h'}^{s+h',x,n} - Y_{s+h'}^{s,x,n})|^2 \rho^{-1}(x) dx] &+ \sup_n E[\int_{s+h'}^\theta \int_{\mathcal{O}} |D_\theta(Y_r^{s+h',x,n} - Y_r^{s,x,n})|^2 \rho^{-1}(x) dx dr] \\
&+ \sup_n E[\int_{s+h'}^\theta \int_{\mathcal{O}} |D_\theta(Z_r^{s+h',x,n} - Z_r^{s,x,n})|^2 \rho^{-1}(x) dx dr] \\
&\leq C_p \sup_n E[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx]. \tag{4.16}
\end{aligned}$$

So we only need to estimate $E[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx]$. Note that by Condition (H.2)',

$$\begin{aligned}
&E[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx] \\
&\leq C_p E[\int_{\mathcal{O}} |X_\theta^{s+h',x} - X_\theta^{s,x}|^2 \rho^{-1}(x) dx] + C_p E[\int_{\mathcal{O}} |Y_\theta^{s+h',x,n} - Y_\theta^{s,x,n}|^2 \rho^{-1}(x) dx] \\
&\quad + C_p E[\int_{s+h'}^\theta \int_{\mathcal{O}} |X_r^{s+h',x} - X_r^{s,x}|^2 (1 + |Y_r^{s+h',x,n}|^{p-1})^2 |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr] \\
&\quad + C_p E[\int_{s+h'}^\theta \int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^2 (1 + |Y_r^{s+h',x,n}|^{p-2} + |Y_r^{s,x,n}|^{p-2})^2 |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr] \\
&\leq C_p E[\int_{\mathcal{O}} |X_\theta^{s+h',x} - X_\theta^{s,x}|^2 \rho^{-1}(x) dx] + C_p E[\int_{\mathcal{O}} |Y_\theta^{s+h',x,n} - Y_\theta^{s,x,n}|^2 \rho^{-1}(x) dx] \\
&\quad + C_p \int_{s+h'}^\theta \sqrt{E[\int_{\mathcal{O}} (|Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4 + |X_r^{s+h',x} - X_r^{s,x}|^4) \rho^{-1}(x) dx]} \\
&\quad \quad \times (E[\int_{\mathbb{R}^d} (1 + |Y_r^{s+h',x,n}|^{8p-8} + |Y_r^{s,x,n}|^{8p-16}) \rho^{-1}(x) dx])^{\frac{1}{4}} \\
&\quad \quad \times (E[\int_{\mathbb{R}^d} |D_\theta Y_r^{s+h',x,n}|^8 \rho^{-1}(x) dx])^{\frac{1}{4}} dr. \tag{4.17}
\end{aligned}$$

From (2.2), we have

$$\begin{aligned}
X_r^{s+h',x} - X_r^{s,x} &= - \int_s^{s+h'} b(X_u^{s,x}) du - \int_s^{s+h'} \sigma(X_u^{s,x}) dW_u \\
&\quad + \int_{s+h'}^r (b(X_u^{s+h',x}) - b(X_u^{s,x})) du + \int_{s+h'}^r (\sigma(X_u^{s+h',x}) - \sigma(X_u^{s,x})) dW_u.
\end{aligned}$$

For $q = 4$ or 8 , applying Itô's formula to $|X_r^{s+h',x} - X_r^{s,x}|^q$, we have

$$E[\int_{\mathbb{R}^d} |X_r^{s+h',x} - X_r^{s,x}|^q \rho^{-1}(x) dx]$$

$$\begin{aligned}
&\leq C_p E\left[\int_{\mathbb{R}^d} h'^{\frac{q}{2}} \left(\int_s^{s+h'} (1 + |X_u^{s,x}|)^2 du\right)^{\frac{q}{2}} \rho^{-1}(x) dx\right] + C_p \int_{\mathbb{R}^d} E\left[\left(\int_s^{s+h'} |\sigma(X_u^{s,x})|^2 du\right)^{\frac{q}{2}}\right] \rho^{-1}(x) dx \\
&\quad + C_p E\left[\int_{s+h'}^r \int_{\mathbb{R}^d} |X_u^{s+h',x} - X_u^{s,x}|^q \rho^{-1}(x) dx du\right] \\
&\leq C_p h'^{\frac{q}{2}} + C_p E\left[\int_{s+h'}^r \int_{\mathbb{R}^d} |X_u^{s+h',x} - X_u^{s,x}|^q \rho^{-1}(x) dx du\right].
\end{aligned}$$

By Gronwall's inequality, we have for $s + h' \leq r \leq T$,

$$E\left[\int_{\mathbb{R}^d} |X_r^{s+h',x} - X_r^{s,x}|^q \rho^{-1}(x) dx\right] \leq C_p h'^{\frac{q}{2}}. \quad (4.18)$$

Similarly, noticing (3.4) and applying Itô's formula to $|Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4$, we have

$$\begin{aligned}
&E\left[\int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4 \rho^{-1}(x) dx\right] \\
&\quad + 6E\left[\int_r^T \int_{\mathcal{O}} |Y_u^{s+h',x,n} - Y_u^{s,x,n}|^2 |Z_u^{s+h',x,n} - Z_u^{s,x,n}|^2 \rho^{-1}(x) dx du\right] \\
&\leq Lh'^2 + (6L + 12L^2)E\left[\int_r^T \int_{\mathcal{O}} |Y_u^{s+h',x,n} - Y_u^{s,x,n}|^4 \rho^{-1}(x) dx du\right] \\
&\quad + 12LE\left[\int_r^T \int_{\mathcal{O}} |Y_u^{s+h',x,n} - Y_u^{s,x,n}|^2 (1 + |Y_u^{s,x,n}|^{2p}) |X_u^{s+h',x} - X_u^{s,x}|^2 \rho^{-1}(x) dx du\right] \\
&\quad + 12L^2E\left[\int_r^T \int_{\mathcal{O}} |Y_u^{s+h',x,n} - Y_u^{s,x,n}|^2 |X_u^{s+h',x} - X_u^{s,x}|^2 \rho^{-1}(x) dx du\right] \\
&\leq Lh'^2 + (2\varepsilon + 6L + 12L^2)E\left[\int_r^T \int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4 \rho^{-1}(x) dx du\right] \\
&\quad + C_p \sqrt{\sup_n E\left[\int_r^T \int_{\mathbb{R}^d} (1 + |Y_u^{s,x,n}|^{8p}) \rho^{-1}(x) dx du\right] E\left[\int_r^T \int_{\mathcal{O}} |X_r^{s+h',x} - X_r^{s,x}|^8 \rho^{-1}(x) dx du\right]} \\
&\quad + C_p E\left[\int_r^T \int_{\mathcal{O}} |X_r^{s+h',x} - X_r^{s,x}|^4 \rho^{-1}(x) dx du\right].
\end{aligned}$$

Therefore, we can deduce from Lemma 3.4, (4.18) and Gronwall's inequality that, for $s + h' \leq r \leq T$,

$$E\left[\int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4 \rho^{-1}(x) dx\right] \leq C_p h'^2. \quad (4.19)$$

By (4.18), (4.19), Lemmas 3.4 and 4.2, we know from (4.17) that

$$\sup_n E\left[\int_{\mathcal{O}} |J(s, s + h')|^2 \rho^{-1}(x) dx\right] \leq C_p h'. \quad (4.20)$$

Therefore, by (4.16) and (4.20) we have

$$\begin{aligned}
&\sup_n E\left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta} Y_{s+h'}^{s+h',x,n} - D_{\theta} Y_{s+h'}^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds\right] \\
&\leq C_p \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \sup_n E\left[\int_{\mathcal{O}} |D_{\theta} Y_{s+h'}^{s+h',x,n} - D_{\theta} Y_{s+h'}^{s,x,n}|^2 \rho^{-1}(x) dx\right] d\theta ds \leq C_p h'.
\end{aligned} \quad (4.21)$$

Finally, by (4.6), (4.13)–(4.15) and (4.21), (4ii) is satisfied. Theorem 4.3 is proved. \diamond

From the relative compactness of u_n in $L^2(\Omega \times [0, T]; L^2_\rho(\mathcal{O}; \mathbb{R}^1))$ for a bounded domain \mathcal{O} in \mathbb{R}^d , we can further prove that there exists a subsequence of u_n , still denoted by u_n , which converges strongly in $L^2(\Omega \times [0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$. We start from an easy lemma.

Lemma 4.4 *Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), for $u_n(t, x)$ defined in Theorem 4.3, we have $\sup_n E[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^{2p} \rho^{-1}(x) dx ds] < \infty$. Furthermore,*

$$\lim_{N \rightarrow \infty} \sup_n E[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)| I_{U_N^c}(x) \rho^{-1}(x) dx ds] = 0,$$

where $U_N = \{x \in \mathbb{R}^d : |x| \leq N\}$.

Proof. The claim $\sup_n E[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^{2p} \rho^{-1}(x) dx ds] < \infty$ follows immediately from the equivalence of norm principle Lemma 2.7 and Lemma 3.4. Let's prove the second part of this lemma. Since $\int_{\mathbb{R}^d} \rho^{-1}(x) dx < \infty$, the claim follows from the following inequality

$$\begin{aligned} & \sup_n E[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^2 I_{U_N^c}(x) \rho^{-1}(x) dx ds] \\ & \leq \left(\sup_n E[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^{2p} \rho^{-1}(x) dx ds] \right)^{\frac{1}{p}} \left(E[\int_0^T \int_{\mathbb{R}^d} |I_{U_N^c}(x)|^{\frac{p}{p-1}} \rho^{-1}(x) dx ds] \right)^{\frac{p-1}{p}}. \end{aligned}$$

\diamond

Theorem 4.5 *Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), if $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the solution of BSDEs (3.4) and $Y_s^{t,x}$ is the weak limit of $Y_s^{t,x,n}$ in $L^2_\rho(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$, then there is a subsequence of $Y_s^{t,x,n}$, still denoted by $Y_s^{t,x,n}$, converging strongly to $Y_s^{t,x}$ in $L^2(\Omega \times [t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^1))$.*

Proof. By Theorem 4.3, we know that for each bounded domain $\mathcal{O} \subset \mathbb{R}^d$, there exists a subsequence of u_n which converges strongly in $L^2(\Omega \times [0, T]; L^2_\rho(\mathcal{O}; \mathbb{R}^1))$. So for U_1 , we are able to extract a subsequence from $u_n(s, x)$, denoted by $u_{1n}(s, x)$, which converges strongly in $L^2(\Omega \times [0, T]; L^2_\rho(U_1; \mathbb{R}^1))$. Obviously the subsequence $u_{1n}(s, x)$ still satisfies the conditions in Theorem 4.3. Applying Theorem 4.3 again, we are able to extract another subsequence from $u_{1n}(s, x)$, denoted by $u_{2n}(s, x)$, that converges strongly in $L^2(\Omega \times [0, T]; L^2_\rho(U_2; \mathbb{R}^1))$. Actually we can do this procedure for all U_i , $i = 1, 2, \dots$. Now we pick up the diagonal sequence $u_{ii}(s, x)$, $i = 1, 2, \dots$, and still denote this sequence by u_n for convenience. It is easy to see that u_n converges strongly in all $L^2(\Omega \times [0, T]; L^2_\rho(U_i; \mathbb{R}^1))$, $i = 1, 2, \dots$. For arbitrary $\varepsilon > 0$, noticing Lemma 4.4, we can find $j(\varepsilon)$ large enough s.t.

$$\sup_n E[\int_0^T \int_{U_{j(\varepsilon)}^c} |u_n(s, x)|^2 \rho^{-1}(x) dx ds] < \frac{\varepsilon}{8}.$$

For this $j(\varepsilon)$, there exists $n^*(\varepsilon) > 0$ s.t. when $m, n \geq n^*(\varepsilon)$, we know

$$\|u_m - u_n\|_{L^2(\Omega \times [0, T]; L^2_\rho(U_{j(\varepsilon)}; \mathbb{R}^1))}^2 = \int_0^T \int_{U_{j(\varepsilon)}} |u_m(s, x) - u_n(s, x)|^2 \rho^{-1}(x) dx ds < \frac{\varepsilon}{2}.$$

Therefore as $m, n \geq n^*(\varepsilon)$,

$$\begin{aligned} \|u_m - u_n\|_{L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))}^2 &\leq E\left[\int_0^T \int_{U_{j(\varepsilon)}} |u_m(s, x) - u_n(s, x)|^2 \rho^{-1}(x) dx ds\right] \\ &\quad + E\left[\int_0^T \int_{U_{j(\varepsilon)}^c} (2|u_m(s, x)|^2 + 2|u_n(s, x)|^2) \rho^{-1}(x) dx ds\right] < \varepsilon. \end{aligned}$$

That is to say u_n converges strongly in $L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$. Then the strong convergence of $Y^{t, \cdot, n}$ follows from the standard equivalence of norm principle argument. On the other hand, $Y_s^{t, x, n}$ is also weakly convergent in $L^2(\Omega \times [t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ with the weak limit $Y_s^{t, x}$. Therefore $Y_s^{t, x, n}$ converges strongly to $Y_s^{t, x}$ in $L^2(\Omega \times [t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$. \diamond

Considering the strongly convergent subsequence $\{Y^{t, \cdot, n}\}_{n=1}^\infty$ derived from Theorem 4.5 and using B-D-G inequality to BDSDE (3.4), we can prove that for arbitrary m, n ,

$$\begin{aligned} &E\left[\sup_{s \in [t, T]} \int_{\mathbb{R}^d} |Y_s^{t, x, m} - Y_s^{t, x, n}|^2 \rho^{-1}(x) dx\right] + E\left[\int_t^T \int_{\mathbb{R}^d} |Z_r^{t, x, m} - Z_r^{t, x, n}|^2 \rho^{-1}(x) dx dr\right] \\ &\leq C_p \left(E\left[\int_t^T \int_{\mathbb{R}^d} |Y_r^{t, x, m} - Y_r^{t, x, n}|^2 \rho^{-1}(x) dx dr\right]\right. \\ &\quad \times E\left[\int_t^T \int_{\mathbb{R}^d} (1 + |f_0(r, x)|^2 + |Y_r^{t, x, n}|^{2p} + |Y_r^{t, x, m}|^{2p}) \rho^{-1}(x) dx dr\right]^{\frac{1}{2}} \\ &\quad \left.+ C_p E\left[\int_t^T \int_{\mathbb{R}^d} |g(r, X_r^{t, x}, Y_r^{t, x, m}) - g(r, X_r^{t, x}, Y_r^{t, x, n})|^2 \rho^{-1}(x) dx dr\right]\right). \end{aligned} \quad (4.22)$$

Using strong subsequence of $Y^{t, \cdot, n}$ and the Lipschitz continuity of g , by the dominated convergence theorem we can conclude from (4.22) that this subsequence $\{Y^{t, \cdot, n}\}_{n=1}^\infty$ converges strongly also in $S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ and the corresponding subsequence of $\{Z^{t, \cdot, n}\}_{n=1}^\infty$ converges strongly $M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ as well. Certainly the strong convergence limit should be identified with the weak convergence limit $Z^{t, \cdot}$. Hence the following corollary follows without a surprise.

Corollary 4.6 *Let $(Y^{t, \cdot}, Z^{t, \cdot})$ be the solution of BDSDE (3.7) and $(Y^{t, \cdot, n}, Z^{t, \cdot, n})$ be the subsequence of the solutions of BDSDE (3.4), of which $Y^{t, \cdot, n}$ converges strongly to $Y^{t, \cdot}$ in $L^2(\Omega \times [t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$, then $(Y^{t, \cdot, n}, Z^{t, \cdot, n})$ also converges strongly to $(Y^{t, \cdot}, Z^{t, \cdot})$ in $S^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$.*

As for $Y_s^{t, x}$, we further have

Lemma 4.7 *Under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), $E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t, x}|^{2p} \rho^{-1}(x) dx ds] < \infty$ and $Y_s^{t, x} = Y_s^{s, X_s^{t, x}}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.*

Proof. First by Lemma 2.7 and Corollary 4.6, we have

$$\begin{aligned} &E\left[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t, x} - Y_s^{s, X_s^{t, x}}|^2 \rho^{-1}(x) dx ds\right] \\ &\leq \lim_{n \rightarrow \infty} 2E\left[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t, x, n} - Y_s^{t, x}|^2 \rho^{-1}(x) dx ds\right] + \lim_{n \rightarrow \infty} 2E\left[\int_t^T \int_{\mathbb{R}^d} |Y_s^{s, X_s^{t, x, n}} - Y_s^{s, X_s^{t, x}}|^2 \rho^{-1}(x) dx ds\right] \\ &\leq \lim_{n \rightarrow \infty} 2E\left[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t, x, n} - Y_s^{t, x}|^2 \rho^{-1}(x) dx ds\right] + \lim_{n \rightarrow \infty} C_p E\left[\sup_{s \leq r \leq T} \int_{\mathbb{R}^d} |Y_r^{s, x, n} - Y_r^{s, x}|^2 \rho^{-1}(x) dx\right] = 0. \end{aligned}$$

Hence,

$$Y_s^{t,x} = Y_s^{s, X_s^{t,x}} \text{ for a.e. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.} \quad (4.23)$$

If we define $Y_s^{s,x} = u(s, x)$, then by (4.23) and Lemma 2.7 again we also have

$$\lim_{n \rightarrow \infty} E \left[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x) - u(s, x)|^2 \rho^{-1}(x) dx ds \right] = 0, \quad (4.24)$$

and

$$Y_s^{t,x} = u(s, X_s^{t,x}) \text{ for a.e. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

Therefore, we claim that the strong limit of $u_n(s, x)$ in $L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ is $u(s, x)$.

By the equivalence of norm principle, to get $E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x}|^{2p} \rho^{-1}(x) dx ds] < \infty$, we only need to prove $E[\int_0^T \int_{\mathbb{R}^d} |u(s, x)|^{2p} \rho^{-1}(x) dx ds] < \infty$. For this, we first derive from $\lim_{n \rightarrow \infty} E[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x) - u(s, x)|^2 \rho^{-1}(x) dx ds] = 0$ a subsequence of $\{u_n(s, x)\}_{n=1}^\infty$, still denoted by $\{u_n(s, x)\}_{n=1}^\infty$, s.t.

$$u_n(s, x) \longrightarrow u(s, x) \text{ and } \sup_n |u_n(s, x)|^{2p} < \infty \text{ for a.e. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

By a similar argument as in Lemma 4.4, for this subsequence u_n , we can prove, using Hölder inequality, that for any $\delta > 0$,

$$\lim_{N \rightarrow \infty} \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^{2p-\delta} I_{\{|u_n(s, x)|^{2p-\delta} > N\}}(s, x) \rho^{-1}(x) dx ds \right] = 0.$$

That is to say that $|u_n(s, x)|^{2p-\delta}$ is uniformly integrable. Moreover by the fact that $u_n(s, x) \longrightarrow u(s, x)$ for a.e. $s \in [0, T], x \in \mathbb{R}^d$ a.s. and generalized Lebesgue's dominated convergence theorem [24], we have

$$\begin{aligned} E \left[\int_0^T \int_{\mathbb{R}^d} |u(s, x)|^{2p-\delta} \rho^{-1}(x) dx ds \right] &= \lim_{n \rightarrow \infty} E \left[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^{2p-\delta} \rho^{-1}(x) dx ds \right] \\ &\leq C_p \left(\sup_n E \left[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)|^{2p} \rho^{-1}(x) dx ds \right] \right)^{\frac{2p-\delta}{2p}} \leq C_p, \end{aligned}$$

where the last $C_p < \infty$ is a constant independent of n and δ . Then using Fatou lemma to take the limit as $\delta \rightarrow 0$ in the above inequality, we can get $E[\int_0^T \int_{\mathbb{R}^d} |u(s, x)|^{2p} \rho^{-1}(x) dx ds] < \infty$. \diamond

Indeed, with Corollary 4.6 and Lemma 4.7, using Itô's formula to $e^{Kr} |Y_r^{t,x}|^{2p}$, we can further prove that $Y^{\cdot, \cdot} \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$ (To see similar calculations, one can refer to the proof of Lemma 3.4 in Section 7).

Proposition 4.8 For $(Y^{\cdot, \cdot}, Z^{\cdot, \cdot})$ and $(Y^{\cdot, \cdot, n}, Z^{\cdot, \cdot, n})$ given in Corollary 4.6, $Y^{\cdot, \cdot} \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$.

Now we are ready to prove the identification of the limiting BDSDEs.

Lemma 4.9 The random field U, V, Y and Z have the following relation:

$$U_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}), \quad V_s^{t,x} = g(s, X_s^{t,x}, Y_s^{t,x}) \text{ for a.e. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

Proof. Let \mathcal{K} be a set in $\Omega \times [t, T] \times \mathbb{R}^d$ s.t. $\sup_n |Y_s^{t,x,n}| + \sup_n |Z_s^{t,x,n}| + |f_0(s, X_s^{t,x})| + |g(s, X_s^{t,x}, 0)| < K$. First we can find a subsequence of $(Y_s^{t,x,n}, Z_s^{t,x,n})$, still denoted by $(Y_s^{t,x,n}, Z_s^{t,x,n})$, satisfying

$(Y_s^{t,x,n}, Z_s^{t,x,n}) \rightarrow (Y_s^{t,x}, Z_s^{t,x})$ a.s. and $\sup_n |Y_s^{t,x,n}| + \sup_n |Z_s^{t,x,n}| < \infty$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. Then it turns out that as $K \rightarrow \infty$, $\mathcal{K} \uparrow \Omega \times [t, T] \times \mathbb{R}^d$. Moreover it is easy to see that for this subsequence,

$$\begin{aligned} & E\left[\int_t^T \int_{\mathbb{R}^d} 2(\sup_n |f_n(s, X_s^{t,x}, Y_s^{t,x,n})|^2 + |f(s, X_s^{t,x}, Y_s^{t,x})|^2) I_{\mathcal{K}}(s, x) \rho^{-1}(x) dx ds\right] \\ & \leq C_p E\left[\int_t^T \int_{\mathbb{R}^d} (|f_0(s, X_s^{t,x})|^2 + \sup_n |Y_s^{t,x,n}|^{2p}) I_{\mathcal{K}}(s, x) \rho^{-1}(x) dx ds\right] \\ & \quad + C_p E\left[\int_t^T \int_{\mathbb{R}^d} (|f_0(s, X_s^{t,x})|^2 + |Y_s^{t,x}|^{2p}) I_{\mathcal{K}}(s, x) \rho^{-1}(x) dx ds\right] < \infty. \end{aligned}$$

Thus, we can apply Lebesgue's dominated convergence theorem to the following estimate:

$$\begin{aligned} & \lim_{n \rightarrow \infty} E\left[\int_t^T \int_{\mathbb{R}^d} |f_n(s, X_s^{t,x}, Y_s^{t,x,n}) I_{\mathcal{K}}(s, x) - f(s, X_s^{t,x}, Y_s^{t,x}) I_{\mathcal{K}}(s, x)|^2 \rho^{-1}(x) dx ds\right] \\ & \leq 2E\left[\int_t^T \int_{\mathbb{R}^d} \lim_{n \rightarrow \infty} |f_n(s, X_s^{t,x}, Y_s^{t,x,n}) - f(s, X_s^{t,x}, Y_s^{t,x,n})|^2 I_{\mathcal{K}}(s, x) \rho^{-1}(x) dx ds\right] \\ & \quad + 2E\left[\int_t^T \int_{\mathbb{R}^d} \lim_{n \rightarrow \infty} |f(s, X_s^{t,x}, Y_s^{t,x,n}) - f(s, X_s^{t,x}, Y_s^{t,x})|^2 I_{\mathcal{K}}(s, x) \rho^{-1}(x) dx ds\right]. \end{aligned} \quad (4.25)$$

Since $Y_s^{t,x,n} \rightarrow Y_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s., there exists a $N(s, x, \omega)$ s.t. when $n \geq N(s, x, \omega)$, $|Y_s^{t,x,n}| \leq |Y_s^{t,x}| + 1$. We take $f_n(s, x, y)$ to be the smotherized truncations to $f(s, x, y)$ on variable y , so taking $n \geq \max\{N(s, x, \omega), |Y_s^{t,x}| + 1\}$, we have $f_n(s, X_s^{t,x}, Y_s^{t,x,n}) = f(s, X_s^{t,x}, \frac{\inf(n, |Y_s^{t,x,n}|)}{|Y_s^{t,x,n}|} Y_s^{t,x,n}) + \partial_y f(s, x, \frac{n}{|Y_s^{t,x,n}|} Y_s^{t,x,n})(Y_s^{t,x,n} - \frac{n}{|Y_s^{t,x,n}|} Y_s^{t,x,n}) I_{\{|Y_s^{t,x,n}| > n\}} = f(s, X_s^{t,x}, Y_s^{t,x,n})$. That is to say

$$\lim_{n \rightarrow \infty} |f_n(s, X_s^{t,x}, Y_s^{t,x,n}) - f(s, X_s^{t,x}, Y_s^{t,x,n})|^2 = 0 \text{ for a.e. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

On the other hand, $\lim_{n \rightarrow \infty} |f(s, X_s^{t,x}, Y_s^{t,x,n}) - f(s, X_s^{t,x}, Y_s^{t,x})|^2 = 0$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. is obvious due to the continuity of $y \rightarrow f(s, x, y)$.

Therefore by (4.25), $f_n(s, X_s^{t,x}, Y_s^{t,x,n}) I_{\mathcal{K}}(s, x) = U_s^{t,x,n} I_{\mathcal{K}}(s, x)$ converges strongly to $f(s, X_s^{t,x}, Y_s^{t,x}) I_{\mathcal{K}}(s, x)$ in $L^2_\rho(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$, but $U_s^{t,x,n} I_{\mathcal{K}}(s, x)$ converges weakly to $U_s^{t,x} I_{\mathcal{K}}(s, x)$ in $L^2_\rho(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$, so $f(s, X_s^{t,x}, Y_s^{t,x}) I_{\mathcal{K}}(s, x) = U_s^{t,x} I_{\mathcal{K}}(s, x)$ for a.e. $r \in [t, T]$, $x \in \mathbb{R}^d$ a.s. Taking $K \rightarrow \infty$, we obtain the first part of Lemma 4.9. The other part of Lemma 4.9 can be proved similarly. \diamond

Proof of Theorem 3.2. With Proposition 4.8 and Lemma 4.9, the existence of solution of BDSDE (3.1) is easy to see. The uniqueness can be proved using a standard subtraction, Itô's formula and Gronwall inequality argument. Here the monotonicity plays an important role. \diamond

By the stochastic flow property $X_r^{s, X_s^{t,x}} = X_r^{t,x}$ for $t \leq s \leq r \leq T$ and the uniqueness of solution of BDSDE (3.1), following a similar argument as Proposition 3.4 in [28] we have

Corollary 4.10 *Under the conditions of Theorem 3.2, let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of BDSDE (3.1), then*

$$Y_s^{t,x} = Y_s^{s, X_s^{t,x}}, \quad Z_s^{t,x} = Z_s^{s, X_s^{t,x}} \quad \text{for any } s \in [t, T], \text{ a.e. } x \in \mathbb{R}^d \text{ a.s.}$$

Naturally, we can relate the $S^{2p,0}([t, T], L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T], L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ solution of BDSDE (3.1) to the weak solution of SPDE (2.1) with finite dimensional noise. We have the following theorem:

Theorem 4.11 *Define $u(t, x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of BDSDE (3.1) under Conditions (H.1)–(H.2), (H.3)*, (H.4)–(H.6), then $u(t, x)$ is the unique weak solution of SPDE (2.1) with finite dimensional noise. Moreover, let u be a representative in the equivalence class of the solution of the SPDE (2.1) in $S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$ with $\sigma^* \nabla u \in M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$, then $u(t, x) = Y_t^{t,x}$ for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$ a.s. and*

$$u(s, X_s^{t,x}) = Y_s^{t,x}, \quad (\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.} \quad (4.26)$$

Proof. Using Corollary 4.6, we first prove the relationship between (Y, Z) and u , when we take $u(t, x) = Y_t^{t,x}$. Having proved Lemma 4.7, we only need to prove that $(\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. This can be deduced from (4.26) and the strong convergence of $Z^{\cdot, \cdot, n}$ to $Z^{\cdot, \cdot}$ in $L^2(\Omega \times [t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ by the similar argument as in Proposition 4.2 in [28].

We then prove that $u(t, x)$ defined above is the unique weak solution of SPDE (2.1). We start our proof from smoothed SPDE (3.5). Let $u_n(s, x)$ be the weak solution of SPDE (3.5), then $(u_n, \sigma^* \nabla u_n) \in L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and for an arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$,

$$\begin{aligned} & \int_{\mathbb{R}^d} u_n(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} h(x) \varphi(x) dx - \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u_n)(s, x) (\sigma^* \nabla \varphi)(x) dx ds \\ & - \int_t^T \int_{\mathbb{R}^d} u_n(s, x) \operatorname{div}((b - \tilde{A})\varphi)(x) dx ds \\ & = \int_t^T \int_{\mathbb{R}^d} f_n(s, x, u_n(s, x)) \varphi(x) dx ds - \int_t^T \int_{\mathbb{R}^d} \langle g(s, x, u_n(s, x)) \varphi(x) dx, d^\dagger \hat{B}_s \rangle. \end{aligned} \quad (4.27)$$

We can prove that along a subsequence each term of (4.27) converges weakly to the corresponding term of (2.6) in $L^2(\Omega; \mathbb{R}^1)$. By (4.24), we know that u_n converges strongly to u in $L_\rho^2(\Omega \times [0, T] \times \mathbb{R}^d; \mathbb{R}^1)$, thus u_n also converges weakly. Moreover, note $\sup_{x \in \mathbb{R}^d} |\operatorname{div}((b - \tilde{A})\varphi)(x)| < \infty$ and ρ is a continuous function in \mathbb{R}^d . So it is obvious that in the sense of the weak convergence in $L^2(\Omega; \mathbb{R}^d)$,

$$\lim_{n \rightarrow \infty} \int_t^T \int_{\mathbb{R}^d} u_n(s, x) \operatorname{div}((b - \tilde{A})\varphi)(x) dx ds = \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div}((b - \tilde{A})\varphi)(x) dx ds.$$

Also it is easy to see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u_n)(s, x) (\sigma^* \nabla \varphi)(x) dx ds &= -\frac{1}{2} \int_t^T \int_{\mathbb{R}^d} u(s, x) \nabla(\sigma \sigma^* \nabla \varphi \sigma)(x) \rho(x) \rho^{-1}(x) dx ds \\ &= \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u)(s, x) (\sigma^* \nabla \varphi)(x) dx ds. \end{aligned}$$

Note that $f_n(s, X_s^{t,x}, Y_s^{t,x, n})$ converges weakly to $f(s, X_s^{t,x}, Y_s^{t,x})$ in $L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$. In fact we can use the same procedures to prove that $f_n(s, x, u_n(s, x))$ converges weakly to $f(s, x, u(s, x))$ and $g(s, x, u_n(s, x))$ converges weakly to $g(s, x, u(s, x))$ in $L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$. Then following the proof of BDSDE (3.4) converging weakly to BDSDE (3.7) and taking weak limit here in $L^2(\Omega; \mathbb{R}^1)$, we obtain the weak convergence of three terms:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_t^T \int_{\mathbb{R}^d} f_n(s, x, u_n(s, x)) \varphi(x) dx ds - \lim_{n \rightarrow \infty} \int_t^T \int_{\mathbb{R}^d} \langle g(s, x, u_n(s, x)) \varphi(x) dx, d^\dagger \hat{B}_s \rangle \\
&= \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x)) \varphi(x) dx ds - \int_t^T \int_{\mathbb{R}^d} \langle g(s, x, u(s, x)) \varphi(x) dx, d^\dagger \hat{B}_s \rangle.
\end{aligned}$$

Finally, that for any $t \in [0, T]$, $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} u_n(t, x) \varphi(x) dx = \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx$ in the sense of weak convergence in $L^2(\Omega; \mathbb{R}^1)$ can be deduced from Corollary 4.6:

$$\begin{aligned}
\lim_{n \rightarrow \infty} |E[\int_{\mathbb{R}^d} (u_n(t, x) - u(t, x)) \varphi(x) dx]|^2 &\leq \lim_{n \rightarrow \infty} C_p E[\int_{\mathbb{R}^d} |u_n(t, X_t^{0,x}) - u(t, X_t^{0,x})|^2 \rho^{-1}(x) dx] \\
&\leq \lim_{n \rightarrow \infty} C_p E[\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |Y_t^{0,x,n} - Y_t^{0,x}|^2 \rho^{-1}(x) dx] = 0.
\end{aligned}$$

Here the convergence in the $S^{2p,0}$ space gives us a strong result about the convergence $\int_{\mathbb{R}^d} u_n(t, x) \varphi(x) dx \rightarrow \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx$ in $L^2(\Omega; \mathbb{R}^d)$ uniformly in t as $n \rightarrow \infty$. Therefore we prove that (2.6) is satisfied for all $t \in [0, T]$, hence $u(t, x)$ is a weak solution of SPDE (2.1).

The uniqueness of weak solution of SPDE (2.1) can be derived from the uniqueness of solution of BDSDE (2.3). For this, let u be a weak solution of SPDE (2.1). Define $F(s, x) = f(s, x, u(s, x))$ and $G(s, x) = g(s, x, u(s, x))$. Since u is the solution, so $\int_0^T \int_{\mathbb{R}^d} (|u(s, x)|^{2p} + |(\sigma^* \nabla u)(s, x)|^2) \rho^{-1}(x) dx ds < \infty$ and

$$\begin{aligned}
& E[\int_0^T \int_{\mathbb{R}^d} (|F(s, x)|^2 + |G(s, x)|^2) \rho^{-1}(x) dx ds] \\
&\leq C_p E[\int_0^T \int_{\mathbb{R}^d} (1 + |f_0(s, x)|^2 + |g(s, x, 0)|^2 + |u(s, x)|^{2p}) \rho^{-1}(x) dx ds] < \infty.
\end{aligned}$$

Then we get a SPDE with the generator $(F, G) \in L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times L^2(\Omega \times [0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$. For this generator (F, G) , we claim that $(Y_s^{t,x}, Z_s^{t,x}) \triangleq (u(s, X_s^{t,x}), (\sigma^* \nabla u)(s, X_s^{t,x}))$ solves the following linear BDSDE for a.e. $x \in \mathbb{R}^d$ with probability one:

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T F(r, X_r^{t,x}) dr - \int_s^T \langle G(r, X_r^{t,x}), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle. \quad (4.28)$$

First we use the mollifier to smoothenize (h, F, G) , then we get a smoothened sequence (h^m, F^m, G^m) such that $(h^m(\cdot), F^m(s, \cdot), G^m(s, \cdot)) \rightarrow (h(\cdot), F(s, \cdot), G(s, \cdot))$ in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$. Denote by $u_m(t, x)$ the solution of SPDE on $[0, T]$ with terminal value $h^m(x)$ and generator $(F^m(s, x), G^m(s, x))$ and by $(Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ the solution of BDSDE with terminal value $h^m(X_T^{t,x})$ and generator $(F^m(s, X_s^{t,x}), G^m(s, X_s^{t,x}))$, then following classical results of Pardoux and Peng [22], we have $Z_{t,m}^{t,x} = \sigma^* \nabla u_m(t, x)$, and $Y_{s,m}^{t,x} = u_m(s, X_s^{t,x}) = Y_{s,m}^{s, X_s^{t,x}}$, $Z_{s,m}^{t,x} = \sigma^* \nabla u_m(s, X_s^{t,x}) = Z_{s,m}^{s, X_s^{t,x}}$. But by standard estimates $(Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ is a Cauchy sequence in $M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. By equivalence of norm principle, $u_m(s, x)$ is also a Cauchy sequence in \mathcal{H} , where \mathcal{H} is the set of random fields $\{w(s, x); s \in [0, T], x \in \mathbb{R}^d\}$ such that $(w, \sigma^* \nabla w) \in M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ with the norm $\left\{ E[\int_0^T \int_{\mathbb{R}^d} (|w(s, x)|^2 + |(\sigma^* \nabla w)(s, x)|^2) \rho^{-1}(x) dx ds] \right\}^{\frac{1}{2}} < \infty$. So there exists $u \in \mathcal{H}$ such that $(u_m, \sigma^* \nabla u_m) \rightarrow (u, \sigma^* \nabla u)$ in $M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ due to the completeness of \mathcal{H} . By the equivalence of norm principle again, we know that $(Y_s^{t,x}, Z_s^{t,x}) \triangleq (u(s, X_s^{t,x}), (\sigma^* \nabla u)(s, X_s^{t,x}))$ is the limit of Cauchy

sequence of $(Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$. Now it is easy to pass the limit as $m \rightarrow \infty$ on the BDSDE which $(Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ satisfies and conclude that $(Y_s^{t,x}, Z_s^{t,x})$ is a solution of BDSDE (4.28).

Noting the definition of $F(s, x)$, $G(s, x)$, $Y_s^{t,x}$ and $Z_s^{t,x}$, we have that $(Y_s^{t,x}, Z_s^{t,x})$ solves BDSDE (2.3) for a.e. $x \in \mathbb{R}^d$ with probability one. Moreover, by Lemma 2.7,

$$E\left[\int_t^T \int_{\mathbb{R}^d} (|Y_s^{t,x}|^{2p} + |Z_s^{t,x}|^2) \rho^{-1}(x) dx ds\right] \leq C_p \int_t^T \int_{\mathbb{R}^d} |u(s, x)|^{2p} + |(\sigma^* \nabla u)(s, x)|^2 \rho^{-1}(x) dx ds < \infty.$$

As Proposition 4.8, we can further deduce that $Y_s^{t,\cdot} \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$ and therefore $(Y_s^{t,x}, Z_s^{t,x})$ is a solution of BDSDE (2.3). If there is another solution \hat{u} to SPDE (2.1), then similarly we can find another solution $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ to BDSDE (2.3), where

$$\hat{Y}_s^{t,x} = \hat{u}(s, X_s^{t,x}) \text{ and } \hat{Z}_s^{t,x} = (\sigma^* \nabla \hat{u})(s, X_s^{t,x}).$$

By Theorem 3.2, the solution of BDSDE (2.3) is unique, therefore

$$Y_s^{t,x} = \hat{Y}_s^{t,x} \text{ for a.e. } s \in [t, T], x \in \mathbb{R}^d \text{ a.s.}$$

Especially for $t = 0$,

$$Y_s^{0,x} = \hat{Y}_s^{0,x} \text{ for a.e. } s \in [0, T], x \in \mathbb{R}^d \text{ a.s.}$$

By Lemma 2.7 again,

$$E\left[\int_0^T \int_{\mathbb{R}^d} |u(s, x) - \hat{u}(s, x)|^2 \rho^{-1}(x) dx ds\right] \leq C_p E\left[\int_0^T \int_{\mathbb{R}^d} |Y_s^{0,x} - \hat{Y}_s^{0,x}|^2 \rho^{-1}(x) dx ds\right] = 0.$$

So $u(s, x) = \hat{u}(s, x)$ for a.e. $s \in [0, T]$, $x \in \mathbb{R}^d$ a.s. The uniqueness is proved. The uniqueness implies that for any selection u in the equivalence class of solution of the SPDE (2.1), $u(s, x) = Y_s^{s,x}$ for a.e. $s \in [0, T]$, $x \in \mathbb{R}^d$ a.s. Moreover, noting that $(u(s, X_s^{t,x}), \sigma^* \nabla u(s, X_s^{t,x}))$ solves the BDSDE (2.3) and using the uniqueness of solution of BDSDE (2.3) in the equivalence class, we have (4.26) for any representative Y in the equivalence class of the solution of BDSDE (2.3). \diamond

5 BDSDEs and SPDEs with infinite dimensional noise

In this section, the main tasks are to prove the existence and uniqueness of solution to BDSDE with infinite dimensional noise (2.3) and give the probabilistic representation of SPDE (1.1) with an initial value. Considering BDSDE (2.5), the equivalent form of BDSDE (2.3), we assume Conditions (H.1)–(H.6) except for (H.2) which will be replaced by the following refined condition.

(H.2)*. Assume $\|g(0, 0, 0)\|_{L_{V_0}^2(\mathbb{R}^1)}^2 < \infty$, and there exist constants $L, L_j \geq 0$ with $\sum_{j=1}^\infty L_j^2 < \infty$ s.t. for any $s, s_1, s_2 \in [0, T]$, $x, x_1, x_2 \in \mathbb{R}^d$, $y, y_1, y_2 \in \mathbb{R}^1$,

$$\begin{aligned} |f(s, x_1, y) - f(s, x_2, y)| &\leq L(1 + |y|^p)|x_1 - x_2|, \\ |g_j(s_1, x_1, y_1) - g_j(s_2, x_2, y_2)| &\leq L_j(|s_1 - s_2| + |x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

Moreover, we assume $\partial_y f, \partial_y g_j$ exist and satisfy

$$\begin{aligned}
|\partial_y f(s, x_1, y) - \partial_y f(s, x_2, y)| &\leq L(1 + |y|^{p-1})|x_1 - x_2|, \\
|\partial_y f(s, x, y_1) - \partial_y f(s, x, y_2)| &\leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|, \\
|\partial_y g_j(s, x, y)| &\leq L_j, \\
|\partial_y g_j(s, x, y_1) - \partial_y g_j(s, x, y_2)| &\leq L_j(|x_1 - x_2| + |y_1 - y_2|).
\end{aligned}$$

Remark 5.1 (i) An equivalent transformation of BDSDE (2.3) similarly as in (3.3) allows us to take the monotone constant $\mu = 0$ in Condition (H.3) without losing any generality.

(ii) Similar to (3.2), by Condition (H.2)* we can get

$$\sup_{s \in [0, T]} \int_{\mathbb{R}^d} \left(\sum_{j=1}^{\infty} |g_j(s, x, 0)|^2 \right)^p \rho^{-1}(x) dx < \infty. \quad (5.1)$$

Theorem 5.2 Under Conditions (H.1), (H.2)*, (H.3)–(H.6), BDSDE (2.3) has a unique solution $(Y^{\cdot, \cdot, N}, Z^{\cdot, \cdot, N}) \in S^{2p, 0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2, 0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$.

Proof. For every $N \in \mathbb{N}$, it follows from Theorem 3.2 that BDSDE (2.9) has a unique solution $(Y^{\cdot, \cdot, N}, Z^{\cdot, \cdot, N}) \in S^{2p, 0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2, 0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. We then prove that $(Y^{\cdot, \cdot, N}, Z^{\cdot, \cdot, N})$ is a Cauchy sequence.

First note that if we do a similar estimates on $(Y^{\cdot, \cdot, N}, Z^{\cdot, \cdot, N})$ as Lemma 3.4, by Condition (H.1), (H.4) and (5.1) we can get

$$\begin{aligned}
&\sup_N E \left[\sup_{s \in [t, T]} \int_{\mathbb{R}^d} |Y_s^{t, x, N}|^{2p} \rho^{-1}(x) dx \right] + \sup_N E \left[\int_t^T \int_{\mathbb{R}^d} |Z_s^{t, x, N}|^2 \rho^{-1}(x) dx ds \right] \\
&\leq C_p \int_{\mathbb{R}^d} |h(x)|^{2p} \rho^{-1}(x) dx + C_p \int_t^T \int_{\mathbb{R}^d} |f_0(s, x)|^{2p} \rho^{-1}(x) dx ds \\
&\quad + C_p \int_t^T \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} |g_j(s, x, 0)|^{2p} \rho^{-1}(x) dx ds < \infty.
\end{aligned} \quad (5.2)$$

Then for $M, N \in \mathbb{N}$, $j \in \mathbb{N}$, define

$$\begin{aligned}
(\bar{Y}_r^{t, x, M, N}, \bar{Z}_r^{t, x, M, N}) &= (Y_r^{t, x, M} - Y_r^{t, x, N}, Z_r^{t, x, M} - Z_r^{t, x, N}), \\
\bar{g}_j^{M, N}(r, x) &= g_j(r, X_r^{t, x}, Y_r^{t, x, M}) - g_j(r, X_r^{t, x}, Y_r^{t, x, N}).
\end{aligned}$$

Without losing any generality, assume $M \geq N$. Applying Itô's formula to $e^{Kr} |\bar{Y}_r^{t, x, M, N}|^{2p}$ and noting the monotonicity condition of f and the Lipschitz condition of g_j , we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{Ks} |\bar{Y}_s^{t, x, M, N}|^{2p} \rho^{-1}(x) dx + p(2p-1) \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t, x, M, N}|^{2p-2} |\bar{Z}_r^{t, x, M, N}|^2 \rho^{-1}(x) dx dr \\
&\quad + (K - p(2p-1)) \sum_{j=1}^{\infty} L_j^2 - 2\varepsilon \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t, x, M, N}|^{2p} \rho^{-1}(x) dx dr \\
&\leq C_p \sum_{j=N+1}^M L_j^2 \int_s^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t, x, M}|^{2p} \rho^{-1}(x) dx dr + C_p \int_s^T \int_{\mathbb{R}^d} e^{Kr} \left(\sum_{j=N+1}^M |g_j(r, X_r^{t, x}, 0)|^2 \right)^p \rho^{-1}(x) dx dr \\
&\quad - 2p \sum_{j=1}^N \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t, x, M, N}|^{2p-2} \bar{Y}_r^{t, x, M, N} \bar{g}_j^{M, N}(r, x) \rho^{-1}(x) dx d\hat{\beta}_j(r)
\end{aligned}$$

$$\begin{aligned}
& -2p \sum_{j=N+1}^M \int_s^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,M,N}|^{2p-2} \bar{Y}_r^{t,x,M,N} g_j(r, X_r^{t,x}, Y_r^{t,x,M}) \rho^{-1}(x) dx d\hat{\beta}_j(r) \\
& -2p \int_s^T \left\langle \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x,M,N}|^{2p-2} \bar{Y}_r^{t,x,M,N} \bar{Z}_r^{t,x,M,N} \rho^{-1}(x) dx, dW_r \right\rangle.
\end{aligned} \tag{5.3}$$

Choosing sufficiently large K and taking expectation on both sides of (5.3), by Lemma 2.7, (5.1) and (5.2) we have

$$\begin{aligned}
& \lim_{M,N \rightarrow \infty} E \left[\int_s^T \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x,M,N}|^{2p} \rho^{-1}(x) dx dr \right] + \lim_{M,N \rightarrow \infty} E \left[\int_s^T \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x,M,N}|^{2p-2} |\bar{Z}_r^{t,x,M,N}|^2 \rho^{-1}(x) dx dr \right] \\
& \leq \lim_{M,N \rightarrow \infty} C_p \sum_{j=N+1}^M L_j^2 \sup_N E \left[\int_s^T \int_{\mathbb{R}^d} |Y_r^{t,x,N}|^{2p} \rho^{-1}(x) dx dr \right] \\
& + \lim_{M,N \rightarrow \infty} C_p \int_s^T \int_{\mathbb{R}^d} \left(\sum_{j=N+1}^M |g_j(r, x, 0)|^2 \right)^p \rho^{-1}(x) dx dr = 0.
\end{aligned} \tag{5.4}$$

Considering (5.3) again and applying the B-D-G inequality, by (5.4) we have

$$\begin{aligned}
& \lim_{M,N \rightarrow \infty} E \left[\sup_{s \in [t, T]} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x,M,N}|^{2p} \rho^{-1}(x) dx \right] \\
& \leq \lim_{M,N \rightarrow \infty} C_p E \left[\int_t^T \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x,M,N}|^{2p} \rho^{-1}(x) dx dr \right] \\
& + \lim_{M,N \rightarrow \infty} C_p \sum_{j=N+1}^M L_j^{2p} \sup_N E \left[\int_t^T \int_{\mathbb{R}^d} |Y_r^{t,x,N}|^{2p} \rho^{-1}(x) dx dr \right] \\
& + \lim_{M,N \rightarrow \infty} C_p \sum_{j=N+1}^M \int_t^T \int_{\mathbb{R}^d} |g_j(r, x, 0)|^{2p} \rho^{-1}(x) dx dr \\
& + \lim_{M,N \rightarrow \infty} C_p E \left[\int_t^T \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x,M,N}|^{2p-2} |\bar{Z}_r^{t,x,M,N}|^2 \rho^{-1}(x) dx dr \right] = 0.
\end{aligned} \tag{5.5}$$

The estimates (5.4) and (5.5) imply that $(Y^{t,\cdot,N}, Z^{t,\cdot,N})$ is a Cauchy sequence in $S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. So there exists $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ as the limit of $(Y^{t,\cdot,N}, Z^{t,\cdot,N})$. We then show that $(Y^{t,\cdot}, Z^{t,\cdot})$ is the solution of BDSDE (2.3). According to Definition 2.4, we only need to verify that for a.e. $x \in \mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (2.3). For this, we prove that for arbitrary $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$, the integration form of (2.9) with φ converges to the integration form of (2.3) with φ in $L^1(\Omega; \mathbb{R}^1)$ along a subsequence. Due to the strong convergence of $(Y^{t,\cdot,N}, Z^{t,\cdot,N})$ to $(Y^{t,\cdot}, Z^{t,\cdot})$ in $S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,0}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$, only the convergence of the drift term and the diffusion term w.r.t. \hat{B} are not obvious. In fact, the convergence of the diffusion term w.r.t. \hat{B} in $L^1(\Omega; \mathbb{R}^1)$ can be referred to the proof of Theorem 3.2 in [28]. In what follows, we show the convergence of the drift term of the integration form of (2.9) to the corresponding term of (2.3) in $L^1(\Omega; \mathbb{R}^1)$ along a subsequence. Since for arbitrary $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$ and $0 < \delta < 1$,

$$E \left[\left| \int_s^T \int_{\mathbb{R}^d} (f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})) \varphi(x) dx dr \right| \right]$$

$$\leq C_p (E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} \rho^{-1}(x) dx dr])^{\frac{1}{1+\delta}},$$

we only need to prove that along a subsequence

$$\lim_{N \rightarrow \infty} E[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} \rho^{-1}(x) dx dr] = 0. \quad (5.6)$$

First we will find a subsequence of $Y_s^{t,x,N}$, still denoted by $Y_s^{t,x,N}$, s.t.

$$Y_s^{t,x,N} \longrightarrow Y_s^{t,x} \text{ for a.e. } s \in [t, T], \ x \in \mathbb{R}^d \text{ a.s. and } E[\int_t^T \int_{\mathbb{R}^d} \sup_N |Y_s^{t,x,N}|^{2p} \rho^{-1}(x) dx ds] < \infty.$$

For this, by the strong convergence of $Y_s^{t,\cdot,N}$ to $Y_s^{t,\cdot}$ in $S^{2p,0}([t, T]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$, we may assume without losing any generality that $Y_s^{t,x,N} \longrightarrow Y_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. and extract a subsequence of $Y_s^{t,x,N}$, still denoted by $Y_s^{t,x,N}$, s.t.

$$(E[\int_0^T \int_{\mathbb{R}^d} |Y_s^{t,x,N+1} - Y_s^{t,x,N}|^{2p} \rho^{-1}(x) dx ds])^{\frac{1}{2p}} \leq \frac{1}{2^N}.$$

For any N ,

$$|Y_s^{t,x,N}| \leq |Y_s^{t,x,1}| + \sum_{i=1}^{N-1} |Y_s^{t,x,i+1} - Y_s^{t,x,i}| \leq |Y_s^{t,x,1}| + \sum_{i=1}^{\infty} |Y_s^{t,x,i+1} - Y_s^{t,x,i}|.$$

Then by the triangle inequality of a norm, we have

$$\begin{aligned} & (E[\int_t^T \int_{\mathbb{R}^d} \sup_N |Y_s^{t,x,N}|^{2p} \rho^{-1}(x) dx ds])^{\frac{1}{2p}} \\ & \leq (E[\int_t^T \int_{\mathbb{R}^d} (|Y_s^{t,x,1}| + \sum_{i=1}^{\infty} |Y_s^{t,x,i+1} - Y_s^{t,x,i}|)^{2p} \rho^{-1}(x) dx ds])^{\frac{1}{2p}} \\ & \leq (E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,1}|^{2p} \rho^{-1}(x) dx ds])^{\frac{1}{2p}} + \sum_{i=1}^{\infty} (E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,i+1} - Y_s^{t,x,i}|^{2p} \rho^{-1}(x) dx ds])^{\frac{1}{2p}} \\ & \leq (E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,1}|^{2p} \rho^{-1}(x) dx ds])^{\frac{1}{2p}} + \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty. \end{aligned}$$

Noticing

$$\begin{aligned} & E[\int_s^T \int_{\mathbb{R}^d} \sup_N |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} \rho^{-1}(x) dx dr] \\ & \leq C_p E[\int_s^T \int_{\mathbb{R}^d} (|f_0(r, x)|^{1+\delta} + \sup_N |Y_r^{t,x,N}|^{(1+\delta)p} + |Y_r^{t,x}|^{(1+\delta)p}) \rho^{-1}(x) dx dr] < \infty, \end{aligned}$$

we have

$$\sup_N |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} < \infty \text{ for a.e. } s \in [t, T], \ x \in \mathbb{R}^d \text{ a.s.} \quad (5.7)$$

Therefore, by (5.2) and (5.7)

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sup_N E \left[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} \right. \\
& \quad \times I_{\{|f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} > n\}}(r, x) \rho^{-1}(x) dx dr] \\
& \leq \lim_{n \rightarrow \infty} \sup_N (E \left[\int_s^T \int_{\mathbb{R}^d} |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^2 \rho^{-1}(x) dx dr \right])^{\frac{1+\delta}{2}} \\
& \quad \times (E \left[\int_s^T \int_{\mathbb{R}^d} I_{\{|f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} > n\}}(r, x) \rho^{-1}(x) dx dr \right])^{\frac{1-\delta}{2}} \\
& \leq \left(\sup_N E \left[\int_s^T \int_{\mathbb{R}^d} (|f_0(r, x)|^2 + |Y_r^{t,x,N}|^{2p} + |Y_r^{t,x}|^{2p}) \rho^{-1}(x) dx dr \right] \right)^{\frac{1+\delta}{2}} \\
& \quad \times (E \left[\int_s^T \int_{\mathbb{R}^d} \lim_{n \rightarrow \infty} I_{\{\sup_N |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} > n\}}(r, x) \rho^{-1}(x) dx dr \right])^{\frac{1-\delta}{2}} = 0.
\end{aligned}$$

The above convergence demonstrates that $|f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta}$ is uniformly integrable, which together with the fact that $\lim_{N \rightarrow \infty} |f(r, X_r^{t,x}, Y_r^{t,x,N}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} = 0$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. leads to (5.6). So the existence of solution of BDSDE (2.3) follows. As for the uniqueness proof, it is similar to the uniqueness proof of Theorem 3.2. The proof of Theorem 5.2 is completed. \diamond

With the results on BDSDE (2.3), the results on its corresponding SPDE (2.1) follow.

Theorem 5.3 Define $u(t, x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of BDSDE (2.3) under Conditions (H.1), (H.2)*, (H.3)–(H.6), then $u(t, x)$ is the unique weak solution of SPDE (2.1). Moreover,

$$u(s, X_s^{t,x}) = Y_s^{t,x}, \quad (\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

Proof. Consider BDSDE (2.9) and the following SPDE with finite dimensional noise:

$$u_N(t, x) = h(x) + \int_t^T [\mathcal{L}u_N(s, x) + f(s, x, u_N(s, x))] ds - \sum_{j=1}^N \int_t^T g_j(s, x, u_N(s, x)) d\hat{\beta}_j(s). \quad (5.8)$$

By Theorem 4.11 we know that $Y_t^{t,x,N}$ is the weak solution of SPDE (5.8) and

$$u_N(s, X_s^{t,x}) = Y_s^{t,x,N}, \quad (\sigma^* \nabla u_N)(s, X_s^{t,x}) = Z_s^{t,x,N} \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

The remaining part of the proof is to verify that $u_N(s, x)$ is a Cauchy sequence in \mathcal{H} and its limit $u(s, x)$ is the weak solution of SPDE (2.1). The procedure of these proofs are actually similar to Proposition 4.2 and Theorem 4.3 in [28] where a Lipschitz condition to $f(s, x, y)$ on y rather than polynomial growth condition is assumed. However, the polynomial growth condition in the arguments brings the trouble only when verifying that for arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$ the integration form of the drift term of (5.8) converges to the corresponding term of (2.1) in $L^1(\Omega)$, i.e.

$$\lim_{N \rightarrow \infty} E \left[\left| \int_0^T \int_{\mathbb{R}^d} (f(s, x, u_N(s, x)) - f(s, x, u(s, x))) \varphi(x) dx dr \right| \right] = 0. \quad (5.9)$$

If we know that for arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$ and $\delta < 1$,

$$\lim_{N \rightarrow \infty} E \left[\int_0^T \int_{\mathbb{R}^d} |f(s, x, u_N(s, x)) - f(s, x, u(s, x))|^{1+\delta} \rho^{-1}(x) dx dr \right] = 0, \quad (5.10)$$

then (5.9) follows from Hölder inequality immediately. In fact, noting (5.6) we can prove (5.10) by the strong convergence of $u_N(s, x)$ to $u(s, x)$ in \mathcal{H} , $u(s, X_s^{t,x}) = Y_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. and the equivalence of norm principle. \diamond

Remark 5.4 Consider a simpler form of SPDE (2.1) with coefficients f, g being independent of time variables. If we choose Brownian motion \hat{B} in backward SPDE as the time reversal version of Brownian motion B in SPDE (1.1), i.e. $\hat{B}_s = B_{T-s} - B_T$, $0 \leq s \leq T$, and let u be the weak solution of corresponding backward SPDE, then we can see easily that $v(t) \triangleq u(T-t)$ is the unique weak solution of SPDE (1.1) s.t. $(v, \sigma^* \nabla v) \in L^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times L^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$.

6 Infinite horizon BDSDEs and stationary solutions of SPDEs

In this section, we first consider the infinite horizon BDSDE with polynomial growth coefficients. For this, we assume the previous conditions (H.1), (H.2)*, (H.3) with the following changes:

- (H.7). Change “ $s \in [0, T]$ ” to “ $s \in [0, \infty)$ ” in (H.1).
- (H.8). Change “ $s, s_1, s_2 \in [0, T]$ ” to “ $s, s_1, s_2 \in [0, \infty)$ ” in (H.2)*.
- (H.9). Change “ $\mu \in \mathbb{R}^1$ ” to “ $\mu > 0$ with $2\mu - K - p(2p-1) \sum_{j=1}^\infty L_j > 0$ ”, “ $s \in [0, T]$ ” to “ $s \in [0, \infty)$ ” and “ $\leq \mu|y_1 - y_2|^2$ ” to “ $\leq -\mu|y_1 - y_2|^2$ ” in (H.3).

Remark 6.1 From Conditions (H.7) and (H.8), for any given $K > 0$, we can deduce

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^d} e^{-Ks} (|f(s, x, 0)|^{2p} + (\sum_{j=1}^\infty |g_j(s, x, 0)|^2)^p) \rho^{-1}(x) dx ds \\ & \leq C_p \int_0^\infty \int_{\mathbb{R}^d} e^{-Ks} (|f_0(s, x)|^{2p} + (\sum_{j=1}^\infty L_j^2)^p s^{2p} + (\sum_{j=1}^\infty L_j^2)^p |x|^{2p} + (\sum_{j=1}^\infty |g_j(0, 0, 0)|^2)^p) \rho^{-1}(x) dx ds < \infty. \end{aligned}$$

Then we have the existence and uniqueness theorem for BDSDE (2.8):

Theorem 6.2 Under Conditions (H.5)–(H.9), BDSDE (2.8) has a unique solution $(Y^{\cdot, \cdot}, Z^{\cdot, \cdot}) \in S^{2p, -K} \cap M^{2p, -K}([t, \infty]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2, -K}([t, \infty]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$.

Proof. Here we only prove the existence of solution as the uniqueness is similar to the proof of uniqueness in Theorem 3.2. For the same reason of Remark 2.5, for a.e. x , $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (2.8) is equivalent to that for arbitrary $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$, $(Y_s^{t,x}, Z_s^{t,x})$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-Ks} Y_s^{t,x} \varphi(x) dx &= \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx dr + \int_s^\infty \int_{\mathbb{R}^d} K e^{-Kr} Y_r^{t,x} \varphi(x) dx dr \\ &\quad - \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} g(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx d^\dagger \hat{B}_r \\ &\quad - \int_s^\infty \left\langle \int_{\mathbb{R}^d} e^{-Kr} Z_r^{t,x} \varphi(x) dx, dW_r \right\rangle \quad P - \text{a.s.} \end{aligned} \quad (6.1)$$

For each $n \in \mathbb{N}$, we define a sequence of BDSDEs by setting $h = 0$ and $T = n$ in BDSDE (2.3):

$$Y_s^{t,x,n} = \int_s^n f(r, X_r^{t,x}, Y_r^{t,x,n}) dr - \int_s^n g(r, X_r^{t,x}, Y_r^{t,x,n}) d^\dagger \hat{B}_r - \int_s^n \langle Z_r^{t,x,n}, dW_r \rangle. \quad (6.2)$$

It is easy to verify that BDSDE (6.2) satisfies conditions of Theorem 5.2. Hence, for each n , there exists $(Y_s^{t,\cdot,n}, Z_s^{t,\cdot,n}) \in S^{2p,-K}([t, n]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,-K}([t, n]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the unique solution of BDSDE (6.2). Therefore, for arbitrary $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^1)$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-Ks} Y_s^{t,x,n} \varphi(x) dx &= \int_s^n \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}) \varphi(x) dx dr + \int_s^n \int_{\mathbb{R}^d} K e^{-Kr} Y_r^{t,x,n} \varphi(x) dx dr \\ &\quad - \int_s^n \int_{\mathbb{R}^d} e^{-Kr} g(r, X_r^{t,x}, Y_r^{t,x,n}) \varphi(x) dx d^\dagger \hat{B}_r \\ &\quad - \int_s^n \left\langle \int_{\mathbb{R}^d} e^{-Kr} Z_r^{t,x,n} \varphi(x) dx, dW_r \right\rangle \quad P - \text{a.s.} \end{aligned} \quad (6.3)$$

Let $(Y_s^n, Z_s^n)_{s \geq n} = (0, 0)$. Then $(Y_s^{t,\cdot,n}, Z_s^{t,\cdot,n}) \in S^{2p,-K} \cap M^{2p,-K}([t, \infty); L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. We use a similar argument as in the proof of Theorem 5.1 in [28] to prove that $(Y_s^{t,\cdot,n}, Z_s^{t,\cdot,n})$ is a Cauchy sequence in $S^{2p,-K} \cap M^{2p,-K}([t, \infty); L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Assume without loss of any generality $m \geq n$. On the interval $[n, m]$, by Conditions (H.5), (H.7)–(H.9) we can prove that

$$\begin{aligned} &(2p\mu - K - p(2p-1)) \sum_{j=1}^{\infty} L_j - \varepsilon) E \left[\int_n^m \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^{2p} \rho^{-1}(x) dx dr \right] \\ &+ p(2p-1) E \left[\int_n^m \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^{2p-2} |Z_r^{t,x,m} - Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \\ &\leq C_p \int_n^m \int_{\mathbb{R}^d} e^{-Kr} (|f(r, x, 0)|^{2p} + (\sum_{j=1}^{\infty} |g_j(r, x, 0)|^2)^p) \rho^{-1}(x) dx dr \longrightarrow 0 \quad \text{as } m, n \rightarrow \infty. \end{aligned}$$

By above estimates and B-D-G inequality, we further have

$$E \left[\sup_{n \leq s \leq m} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x,m} - Y_s^{t,x,n}|^{2p} \rho^{-1}(x) dx \right] \longrightarrow 0 \quad \text{as } m, n \rightarrow \infty. \quad (6.4)$$

On the interval $[t, n]$, a similar calculation, together with (6.4), leads to

$$\begin{aligned} &(2p\mu - K - p(2p-1)) \sum_{j=1}^{\infty} L_j) E \left[\int_0^n \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^{2p} \rho^{-1}(x) dx dr \right] \\ &+ p(2p-1) E \left[\int_0^n \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x,m} - Y_r^{t,x,n}|^{2p-2} |Z_r^{t,x,m} - Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \\ &\leq E \left[\int_{\mathbb{R}^d} e^{-Kn} |Y_n^{t,x,m}|^{2p} \rho^{-1}(x) dx \right] \longrightarrow 0 \quad \text{as } m, n \rightarrow \infty, \end{aligned}$$

and

$$E \left[\sup_{t \leq s \leq n} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x,m} - Y_s^{t,x,n}|^{2p} \rho^{-1}(x) dx \right] \longrightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Taking account of calculations on both $[t, n]$ and $[n, m]$ we know that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence in $S^{2p,-K} \cap M^{2p,-K}([t, \infty); L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Let $(Y_s^{t,x}, Z_s^{t,x})$ be the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$, then we show that $(Y_s^{t,x}, Z_s^{t,x})$ is a solution of BDSDE (2.8). We only need to

verify that $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (6.1). For this, we prove that along a subsequence (6.3) converges to (6.1) in $L^1(\Omega; \mathbb{R}^1)$ term by term as $n \rightarrow \infty$. Here we only check the drift term which is of polynomial growth, i.e. we show that along a subsequence, as $n \rightarrow \infty$,

$$E[|\int_s^n \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}) \varphi(x) dx dr - \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx dr|] \rightarrow 0.$$

For this, note that for arbitrary $0 < \delta < 1$,

$$\begin{aligned} & E[|\int_s^n \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x,n}) \varphi(x) dx dr - \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx dr|] \\ & \leq C_p E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} |f(r, X_r^{t,x}, Y_r^{t,x,n}) - f(r, X_r^{t,x}, Y_r^{t,x})|^{1+\delta} \rho^{-1}(x) dx dr] \\ & \quad + C_p E[\int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} (|f_0(r, X_r^{t,x})|^2 + |Y_r^{t,x}|^{2p}) \rho^{-1}(x) dx dr]. \end{aligned} \quad (6.5)$$

Both terms on the right hand side of the above inequality converge to 0 along a subsequence as $n \rightarrow \infty$. The convergence of the first term in (6.5) is not obvious, but can be deduced similarly as the proof of (5.6). After verifying other terms of (6.3) converges to the corresponding terms of (6.1) in $L^1(\Omega; \mathbb{R}^1)$ as $n \rightarrow \infty$, we can see that $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (6.1) and the proof of Theorem 6.2 is completed. \diamond

Remark 6.3 The uniqueness of solution of BDSDE (2.8) implies if (\hat{Y}, \hat{Z}) is another solution, then $Y_s^{t,\cdot} = \hat{Y}_s^{t,\cdot}$ for all $s \geq t$ a.s. and $Z_s^{t,\cdot} = \hat{Z}_s^{t,\cdot}$ for a.e. $s \geq t$ a.s. But we can modify the Z at the measure zero exceptional set of s s.t. $Z_s^{t,\cdot} = \hat{Z}_s^{t,\cdot}$ for all $s \geq t$ a.s.

Consider the case when f and g are time-independent coefficients.

Corollary 6.4 Under Conditions (H.5)–(H.9), BDSDE (2.7) has a unique solution $(Y^{t,\cdot}, Z^{t,\cdot})$ in the space $S^{2p,-K} \cap M^{2p,-K}([t, \infty]; L_p^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^{2,-K}([t, \infty]; L_p^2(\mathbb{R}^d; \mathbb{R}^d))$.

We construct a measurable metric dynamical system through defining a measurable and probability preserving shift operator. Let $\hat{\theta}_t = \theta'_t \circ \theta''_t$, $t \geq 0$, where $\theta'_t, \theta''_t : \Omega \rightarrow \Omega$ are measurable mappings on (Ω, \mathcal{F}, P) defined by

$$\theta'_t \left(\begin{pmatrix} \hat{B} \\ W \end{pmatrix} \right) (s) = \begin{pmatrix} \hat{B}_{s+t} - \hat{B}_t \\ W_s \end{pmatrix}, \quad \theta''_t \left(\begin{pmatrix} \hat{B} \\ W \end{pmatrix} \right) (s) = \begin{pmatrix} \hat{B}_s \\ W_{s+t} - W_t \end{pmatrix}.$$

Then for any $s, t \geq 0$, (i). $P = \hat{\theta}_t P$; (ii). $\hat{\theta}_0 = I$, where I is the identity transformation on Ω ; (iii). $\hat{\theta}_s \circ \hat{\theta}_t = \hat{\theta}_{s+t}$. Also for an arbitrary \mathcal{F} measurable ϕ and $t \geq 0$, set

$$\hat{\theta}_t \circ \phi(\omega) = \phi(\hat{\theta}_t(\omega)).$$

For any $r \geq 0$, applying $\hat{\theta}_r$ to SDE (2.2), by the uniqueness of the solution and a perfection procedure (cf. Arnold [1]) we have

$$\hat{\theta}_r \circ X_s^{t,\cdot} = \theta''_r \circ X_s^{t,\cdot} = X_{s+r}^{t+r,\cdot} \quad \text{for all } r, s, t \geq 0 \text{ a.s.}$$

Firstly, we consider the stationarity of BDSDE (2.7), which is equivalent to

$$\begin{cases} Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}) dr - \int_s^T g(X_r^{t,x}, Y_r^{t,x}) d^\dagger \hat{B}_r - \int_s^T \langle Z_r^{t,x}, dW_r \rangle \\ \lim_{T \rightarrow \infty} e^{-KT} Y_T^{t,x} = 0 \quad \text{a.s.} \end{cases}$$

Theorem 6.5 *Under Conditions (H.5)–(H.9), let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of BDSDE (2.7), then $(Y_s^{t,x}, Z_s^{t,x})$ satisfies for any $t \geq 0$,*

$$\hat{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \hat{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{for all } r \geq 0, s \geq t \text{ a.s.}$$

In particular, for any $t \geq 0$,

$$\theta'_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } r \geq 0 \text{ a.s.} \quad (6.6)$$

The proof is similar to the proof of Theorem 1.7 in [29]. So it is omitted in this paper.

If we regard $Y_t^{t,\cdot}$ as a function of t , (6.6) gives a “crude” stationary property of $Y_t^{t,\cdot}$. To make the “crude” stationary property “perfect”, we need to prove the a.s. continuity of $t \rightarrow Y_t^{t,\cdot}$ to obtain one indistinguishable version of $Y_t^{t,\cdot}$ with “perfect” stationary property w.r.t. $\hat{\theta}$. As the a.s. continuity can be similarly proved if one follows the arguments of Theorem 2.11 in [28], here we leave out the proof. Hence it comes without a surprise that

Theorem 6.6 *Under Conditions (H.5)–(H.9), let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of BDSDE (2.7). Then $Y_t^{t,\cdot}$ satisfies the “perfect” stationary property w.r.t. θ' , i.e.*

$$\theta'_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } t, r \geq 0 \text{ a.s.} \quad (6.7)$$

We can further prove an estimate following the proof of Lemma 3.4.

Lemma 6.7 *Under Conditions (H.5)–(H.9), if $(Y^{t,\cdot}, Z^{t,\cdot})$ is the solution of BDSDE (2.7), then we have*

$$\begin{aligned} & E\left[\sup_{s \in [t, \infty)} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x}|^{8p} \rho^{-1}(x) dx\right] + E\left[\int_t^\infty \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x}|^{8p} \rho^{-1}(x) dx ds\right] \\ & + \sup_n E\left[\int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |Y_s^{t,x}|^{8p-2} |Z_s^{t,x}|^2 \rho^{-1}(x) dx ds\right] + \sup_n E\left[\left(\int_t^\infty \int_{\mathbb{R}^d} e^{-\frac{Kr}{4p}} |Z_s^{t,x}|^2 \rho^{-1}(x) dx ds\right)^{4p}\right] < \infty. \end{aligned}$$

Consider BDSDE (2.7) and its solution $Y_s^{t,\cdot}$ on $[t, T]$. We choose \hat{B} as the time reversal of B from time T' , i.e. $\hat{B}_s = B_{T'-s} - B_{T'}$ for $s \geq 0$. Note that the random variable $Y_{T'}^{T',\cdot}$ is $\mathcal{F}_{T',\infty}^{\hat{B}}$ measurable which is independent of \mathcal{F}_t^W . Changing variable in SPDE (1.1), we can deduce from the Correspondence Theorem 5.3 and Remark 5.4 that $v(t, \cdot) = u(T' - t, \cdot) = Y_{T'-t}^{T'-t,\cdot}$ is a weak solution of SPDE (1.1) on $[0, T']$ if $Y_{T'}^{T',\cdot}$ satisfies Condition (H.4). Note $Y_T^{t,x} = Y_T^{T',X_T^{t,x}}$, so Condition (H.4) reads as

$$(H.4)^*. \quad E\left[\int_{\mathbb{R}^d} |Y_{T'}^{T',x}|^{8p} \rho^{-1}(x) dx\right] < L(T') \text{ and } E\left[\int_{\mathbb{R}^d} |Y_{T'}^{t',x} - Y_{T'}^{t,x}|^q \rho^{-1}(x) dx\right] \leq L(T') |t' - t|^{\frac{q}{2}} \text{ for } 2 \leq q \leq 8p \text{ and } X \text{ defined in (2.2), where } L(T') \text{ is a constant which can depend on } T'.$$

Lemma 6.8 *Let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of BDSDE (2.7). Then for arbitrary T' , $Y_{T'}^{T',x}$ satisfies Condition (H.4)*.*

Proof. It follows immediately from Lemmas 2.7 and 6.7 that $E\left[\int_{\mathbb{R}^d} |Y_{T'}^{T',x}|^{8p} \rho^{-1}(x) dx\right] \leq L(T')$. The proof of $E\left[\int_{\mathbb{R}^d} |Y_{T'}^{t',x} - Y_{T'}^{t,x}|^q \rho^{-1}(x) dx\right] \leq L(T') |t' - t|^{\frac{q}{2}}$ is similar to Lemma 6.2 in [28]. \diamond

We can further prove a claim that $v(t, \cdot) = Y_{T'-t}^{T'-t, \cdot}$ does not depend on the choice of T' as done in [28], [29]. On the probability space (Ω, \mathcal{F}, P) , we define $\theta_t : \Omega \rightarrow \Omega$, $t \in \mathbb{R}^1$, as the shift operator of Brownian motion B :

$$\theta_t \circ B_s = B_{s+t} - B_t,$$

then θ satisfies the usual conditions: (i). $P = P \circ \theta_t$; (ii). $\theta_0 = I$; (iii). $\theta_s \circ \theta_t = \theta_{s+t}$. Noticing that \hat{B} is chosen as the time reversal of B and B, W are independent, we can define $\hat{\theta}$, served as the shift operator of \hat{B} and W , to be $\hat{\theta}_t \triangleq (\theta_t)^{-1} \circ \theta_t''$, $t \geq 0$. Actually B is a two-sided Brownian motion, so $(\theta_t)^{-1} = \theta_{-t}$ is well defined (see [1]) and it is easy to see that $\theta_t' \triangleq (\theta_t)^{-1}$, $t \in \mathbb{R}^1$, is a shift operator of \hat{B} .

Since $v(t, \cdot) = u(T' - t, \cdot) = Y_{T'-t}^{T'-t, \cdot}$ a.s., so by (6.7),

$$\theta_r v(t, \cdot, \omega) = \theta'_{-r} u(T' - t, \cdot, \hat{\omega}) = \theta'_{-r} \theta'_r u(T' - t - r, \cdot, \hat{\omega}) = u(T' - t - r, \cdot, \hat{\omega}) = v(t + r, \cdot, \omega),$$

for all $r \geq 0$ and $T' \geq t + r$ a.s. In particular, let $Y(\cdot, \omega) = v(0, \cdot, \omega) = Y_{T'}^{T', \cdot}(\hat{\omega})$, then the above formula implies:

$$\theta_t Y(\cdot, \omega) = Y(\cdot, \theta_t \omega) = v(t, \cdot, \omega) = v(t, \cdot, \omega, v(0, \cdot, \omega)) = v(t, \cdot, \omega, Y(\cdot, \omega)) \text{ for all } t \geq 0 \text{ a.s.}$$

That is to say $v(t, \cdot, \omega) = Y(\cdot, \theta_t \omega) = Y_{T'-t}^{T'-t, \cdot}(\hat{\omega})$ is a stationary solution of SPDE (1.1) w.r.t. θ . Therefore we obtain

Theorem 6.9 *Under Conditions (H.5)–(H.9), for arbitrary T' and $t \in [0, T']$, let $v(t, \cdot) \triangleq Y_{T'-t}^{T'-t, \cdot}$, where $(Y^{t, \cdot}, Z^{t, \cdot})$ is the solution of BDSDE (2.7) with $\hat{B}_s = B_{T'-s} - B_{T'}$ for all $s \geq 0$. Then $v(t, \cdot)$ is a "perfect" stationary solution of SPDE (1.1) independent of the choice of T' .*

It is not difficult to see that in the proof of Theorem 6.2, there is no need to take $h = 0$. In fact we can consider BDSDE (2.3) with an arbitrary h satisfying Condition (H.4). Its solution is denoted by $Y_s^{t, x, T}(h)$. Then under the same conditions as in Theorem 6.9, following the same procedure of this section, we can prove without real difficulty that $Y^{t, \cdot, T}(h) \rightarrow Y^{t, \cdot}$ as $T \rightarrow \infty$ in $S^{2p, -K} \cap M^{2p, -K}([t, \infty]; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$ and $Y^{t, \cdot}$ is the solution of infinite horizon BDSDE (2.7). This implies that $Y_t^{t, \cdot, T}(h) \rightarrow Y_t^{t, \cdot}$ as $T \rightarrow \infty$ in $L^{2p}(\Omega; L_\rho^{2p}(\mathbb{R}^d; \mathbb{R}^1))$. In particular, from previous result of this section, we have

$$Y_{T'-t}^{T'-t, \cdot, T}(h) \rightarrow Y_{T'-t}^{T'-t, \cdot} = v(t, \cdot) = Y(\theta_t \cdot) \text{ and } Y_{T'}^{T', \cdot, T}(h) \rightarrow Y_{T'}^{T', \cdot} = Y(\cdot).$$

Noticing the correspondence of the solution of BDSDE (2.3) and the solution of the SPDE (2.1) in Theorem 5.3, we have $Y_{T'}^{T', \cdot, T}(h, \hat{\omega}) = u(T', h, \hat{\omega})$. Note the correspondence of the forward SPDE (1.1) and the backward SPDE (2.1). Denote $\hat{\omega}^T$ the time reversal Brownian motion of B at time T . Then

$$v(T - T', h, \theta_{-(T-T')} \omega) = u(T - (T - T'), h, \widehat{\theta_{-(T-T')} \omega}^T) = u(T', h, \hat{\omega}) = Y_{T'}^{T', \cdot, T}(h)$$

since

$$\begin{aligned} (\widehat{\theta_{-(T-T')} \omega}^T)(s) &= (\theta_{-(T-T')} \omega)(T - s) - (\theta_{-(T-T')} \omega)(T) \\ &= \omega(-(T - T') + T - s) - \omega(-(T - T') + T) = \omega(T' - s) - \omega(T') = \hat{\omega}(s). \end{aligned}$$

Therefore as $T \rightarrow \infty$, $v(T - T', h, \theta_{-(T-T')}\cdot) = Y_{T'}^{T', T}(h) \rightarrow Y(\cdot)$ in $L^{2p}(\Omega; L^2_p(\mathbb{R}^d; \mathbb{R}^1))$. The result does not depend on the choice of T' . So we have proved

Theorem 6.10 *Assume all conditions in Theorem 6.9 and h satisfies Condition (H.4). Then as $T \rightarrow \infty$, $v(T, h, \theta_{-T}\omega) \rightarrow Y(\cdot)$ in $L^{2p}(\Omega; L^2_p(\mathbb{R}^d; \mathbb{R}^1))$, and $Y(\theta_t\omega)$ is the stationary solution of the SPDE (1.1).*

Remark 6.11 *The result in Theorem 6.10 is also valid under the conditions in [28] and [29] respectively.*

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